# Categories and Functors, Lecture Notes

## 1.1: Intro to Categories

- **Definition:** A category consists of a collection of **objects** and a collection of **morphisms** such that
  1. Each morphism has specified domain and codomain objects, i.e. $f : X \to Y$ indicates a morphism $f$ has a domain $X$ and codomain $Y$
  2. Each object has a designated identity morphism, which can be written as $1_X : X \to X$.
  3. For any composable pair of morphisms it holds that

  $$f : X \to Y, \quad g : Y \to Z \qquad \implies \qquad gf : X \to Z$$

  This is subject to **two important axioms**

  - There is some identity morphism such that $1_Y f$ and $f 1_X$ is equivalent to $f$.
  - Associativity: For any composable triple of morphisms we have that $hgf = h(gf) = (hg)f$.

  A note on the categories we will be working with in this seminar, the focus will be set on **locally small categories**.

  **Definition:** A category is called locally small if between any pair of objects there is only a set's worth of morphisms. I.e. you can represent all the morphisms between two objects as a set. We write $C(X, Y)$ or $\mathrm{Hom}(X, Y)$ in order to denote the set of morphisms from $X$ and $Y$ in a locally small category.

- **Examples of categories**
  - **Set** is a category has sets as objects and are morphisms functions with specified domain and codomain. This includes all possible sets.

- **Group** has groups has objects and group homomorphisms as morphisms.
  - A group $G$ defines a category $\mathrm{B}G$ with a single object. The group's elements are its morphisms
  - For a more concrete example let's consider the integers. The integers is a single object with morphisms corresponding to the with composition corresponding to addition so we have some arbitrary object, call it $*$. And each integer corresponds to a unique morphism from that object to itself. So $1 : * \to *$. Composition between two morphisms in this case corresponds to addition, $g \circ h := g + h$
- **Ring** includes all unital rings and ring homomorphisms.

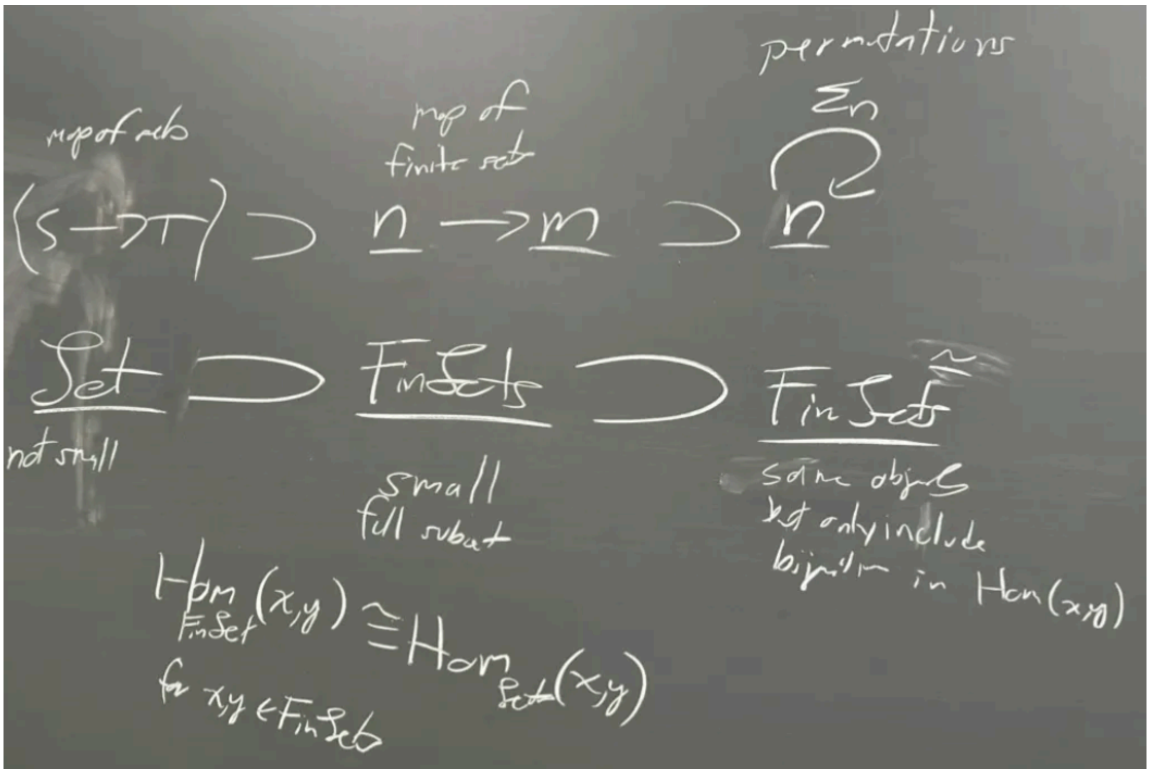
  (of note all of the above are **concrete categories**)

- Definition: An isomorphism is a category of morphism $f : X \to Y$ for which there exists some morphism $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$. Two objects are considered more or less "the same" (imprecisely) when they are isomorphic.
- **Examples of isomorphisms:**
  - Within **Set** isomorphisms are bijective functions
  - Within **Group** isomorphisms are the more familiar term of being bijective homomorphisms. Same goes for **Ring** and **Field.**
- Definition: A **subcategory** denoted by $\mathrm{D}$ is defined by restricting to a subcollection of objects and morphisms (with the idea that the morphisms should remain closed to the objects contained within the subcategory).
  1. The subcategory must contain the domain and codomain of any morphism in the subcategory
  2. The identity morphism for every object in $\mathrm{D}$ must also be in $\mathrm{D}$.
  3. This must also be closed under composition, such that if $f$ and $g$ are two morphisms in $\mathrm{D}$ then $gf$ must also be a morphism in $\mathrm{D}$.

  Note this does not require that you have a full set of morphisms from the full category, only that the morphisms that you do bring in are the identity morphisms and that the objects contained within are the domains and codomains of the morphisms you do bring in. This brings us to another notion:

Definition: full subcategory is a subcategory that includes **all of the morphisms** that were between the objects in the category it derives from. This is not always necessarily the case.

An example of this is Set to Finite Sets, to Finite Sets (only bijections)




## 1.2: Duality

- **Definition:** Allow $\mathrm{C}$ to be any category, the opposite category is the category which possesses the same objects an a morphism $f^{\mathrm{op}}$ in $\mathrm{C}$ for every $f$ in $\mathrm{C}$ such that the domain of $f^{\mathrm{op}}$ is defined to be the codomain of $f$ and the codomain is defined to be the domain.

$$f^{\mathrm{op}} : X \to Y \in \mathrm{C}^{\mathrm{op}}$$
$$f : Y \to X \in \mathrm{C}$$

- Of note, a pair of morphisms $f^{\text{op}}, g^{\text{op}}$ in $\mathrm{C}^{\text{op}}$ is composable precisely when the pair $g, f$ is composable in $\mathrm{C}$.

$$
\begin{array}{ccccc}
f^{\text{op}}: X \to Y,\ g^{\text{op}}: Y \to Z & \in \mathsf{C}^{\text{op}} & \rightsquigarrow & g^{\text{op}} f^{\text{op}}: X \to Z & \in \mathsf{C}^{\text{op}} \\
\updownarrow & & & \updownarrow & \\
g: Z \to Y,\ f: Y \to X & \in \mathsf{C} & \rightsquigarrow & fg: Z \to X & \in \mathsf{C}
\end{array}
$$

- The opposite category carries the same information as in the category $\mathrm{C}$

- A consequence of this opposite category is the following: Any theorem containing a universal qualification of the form "for all categories $\mathrm{C}$" also necessarily applies to the opposites of those categories. You effectively get a **dual theorem** which is a theorem you get from the original proof but the arrows point in the "opposite direction"
  - So **duality** says that any proof in category theory proves two theorems, the original statement and its dual. An example to illustrate this idea. The following statements are equivalent. Say

$$f : x \to y \text{ is an isomorphism in } \mathrm{C}$$

    This is equivalent to

    i. For all objects $c \in \mathrm{C}$, post composition with $f$ (or $f \circ a$) where $a : c \to x$. Leads to a bijection

$$f_* : C(c, x) \to C(c, y)$$

    ii. For all objects $c \in \mathrm{C}$, pre-composition with $f$ or $(a \circ f)$ where $a : y \to c$ defines a bijection

$$f^* : C(y, c) \to C(x, c)$$

  - We can prove an equivalence between the first statement,  and due to dual theory the second statement immediately applies.
    - Assuming that $f : x \to y$ is an isomorphism with some inverse function $g : y \to x$. We can use post composition with $a$ to construct

an inverse function $g_*$ in the sense that

$$g_* : \mathrm{C}(c, y) \to \mathrm{C}(c, x)$$

and

$$g_* f_* : \mathrm{C}(c, x) \to \mathrm{C}(c, x) \quad \text{and} \quad f_* g_* : \mathrm{C}(c, y) \to \mathrm{C}(c, y)$$

Which are identity functions for some $c \to x$ and $c \to y$

- **Definition:** A monomorphism $f$ is defined as a morphism such that for any parallel morphisms $h, k : w \rightrightarrows x$. I.e. two morphisms that have the same domain and codomain, $fh = fk \implies h = k$.
    - **The monomorphisms in the category of Sets are the injective morphisms.**
    - If $f$ was not injective then you can imagine why this would be false. Say $h$ maps some element $a \in w$ to $b$ and $k$ maps that element to $c$. If $f$ isn't injective, i.e maps both of those elements to $d$ in some set that $f$ maps to, then $h \neq k$ but $fh = fk$. Provided that $f$ is injective we have that if $h = k$ then $f(h) = f(k)$. Therefore the expression $fh = fk$ would only hold if $h = k$ and therefore if $f$ is injective $h = k$.
- **Definition:** A **epimorphism** is a morphism under similar conditions say where $hf = kf \implies h = k$ where $h, k : y \rightrightarrows z$.
    - **The epimorphisms in the category of Sets are the surjective morphisms.**
    - If a function $f$ is not surjective then the set it maps to which in this case is $y$, is not fully covered, i.e. there is some element in the output set, say $m$ that does not get mapped to by $f$. Now consider that in the mappings $h, k$, one maps $m$ to $a$ and another maps $m$ to $b$, but all other mappings are consistent. Then for any given input of $f$ we have that $hf = kf$ but from our construction we know $h \neq k$. So an epimorphism has to be "surjective" in this sense.
- **Definition:** functor (covariant functor) **is a morphism** between categories which consists of the following data

- An object $Fc \in D$ for each object $c \in C$
- And a morphism $Ff : Fc \to Fc' \in D$ for each morphism $f \in C$. So that the domain and codomain of the morphism are respectively equal to $F$ applied to the codomain of $f$.

A functor follows two **functorality axioms:**

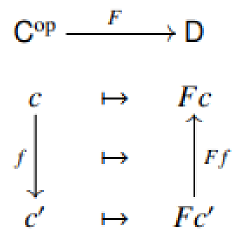
1. For any composable pair $f, g$ in $C$ we also have composition $Fg \cdot Ff = F(g \cdot f)$.
2. For each object $c$ we also have the identity $F(1_c) = 1_{F_c}$.

A functor is therefore the mapping of objects and mapping of morphisms that preserves the structure of a category: in particular, the domains, codomains, composition, and identities.

**Examples:**

- The **endofunctor** is the functor from $P : \mathrm{Set} \to \mathrm{Set}$ which sends a set $A$ to its power set $PA$ or the set of all of the subsets of $A$ and the function $f : A \to B$ which sends $A'$ to $f(A') \subset B$.
- We mentioned prior the broader class of **forgetful functors.** Which is a general term that is used for any functor that forgets structure. For example we can have $U : \mathrm{Group} \to \mathrm{Set}$ which sends a group to its underlying set and homomorphisms to the underlying function.

- **Definition:** a **contravariant functor** is the functor $\mathrm{C}^{\mathrm{op}} \to \mathrm{D}$ which explicitly consists of
  - An object $Fc \in D$, for each object $c \in C$
  - A morphism $Ff : Fc' \to Fc \in D$. So that for each morphism, the domain and codomain of $Ff$ are respectively, equal to $F$ applied to the codomain or domain of $f$. These functors must also satisfy the axioms of being composability and mapping the identities to the respective identities for the objects.




- As a more concrete example of this, the contravariant power set functor $P : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ sends a set to its power set and the function to the inverse image function $f^{-1} : PB \to PA$ that sends $B' \subset B$ and to $f^{-1}(B) \subset A$

- **Lemma: Functors preserve isomorphisms**
  - Consider some functor $F : C \to D$ and an isomorphism $f : x \to y$ with some inverse function $g : y \to x$. Applying the two functorality axioms we get:

$$F(g)F(f) = F(gf) = F(1_x) = 1_{Fx}$$

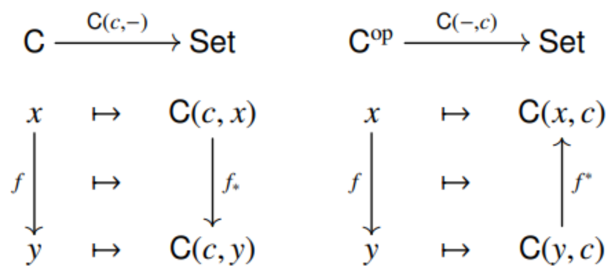
  - Thus $Fg : Fy \to Fx$ is a left inverse to $Ff : Fx \to Fy$. You can exchange the roles of $f$ and $g$ in order to show that it is also a right inverse. Therefore $Ff$ is a bijective homomorphism.
    - **NOT to mention in lecture but something to have in the notes**
      - Example 1.3.9: Let $G$ be a group, regarded as a one-object category $\mathrm{B}G$. A functor $X : \mathrm{B}G \to \mathrm{C}$ specifies an object $X \in \mathrm{C}$. together with an endomorphism $g_* : X \to X$ for each $g \in G$. This assignment must satisfy two conditions.

        i. $h_* g_* = (hg)_*$ for all $g, h \in G$

        ii. $e_* = 1_X$, where $e \in G$ is the identity element.

        This functor defines an action of the group on $\mathrm{B}G \to \mathrm{C}$ is sometimes called a left action. A right action. Is the functor $\mathrm{B}G^{\mathrm{op}} \to \mathrm{C}$

- **Definition 1.3.11:** If $C$ is **locally small** then for any object $c \in \mathrm{C}$ we can define a pair of covariant and contravariant functors




- The notation suggests the action on objects: the functor $\mathrm{C}(c, -)$ carries $x \in \mathrm{C}$ to the set $\mathrm{C}(c, x)$ of arrows from $c$ to $x$ in $\mathrm{C}$. Dually, the functor $\mathrm{C}(-, c)$ carries $x \in C$ to the set $C(x, c)$.
- The $-$ sign is a placeholder for some other sets. When both variables are filled it represents a morphism but with one of the objects set to be variable this represents a either a covariant or contravariant functor.

- **Definition:** If $\mathrm{C}$ is locally small then there is a two-sided represented functor
  - in which a pair of objects $(x, y)$ is mapped to the hom set $\mathrm{C}(x, y)$ and a pair of morphisms $f : w \to x$ and $h : y \to z$ is sent to the function
- We can define

$$\mathsf{C}(x,y) \xrightarrow{(f^*,h_*)} \mathsf{C}(w,z)$$

$$g \quad \mapsto \quad hgf$$

that takes an arrow $g\colon x \to y$ and then pre-composes with $f$ and post-composes with $h$ to obtain $hgf\colon w \to z$.