Fourier Transform on LCA Groups

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Abstract

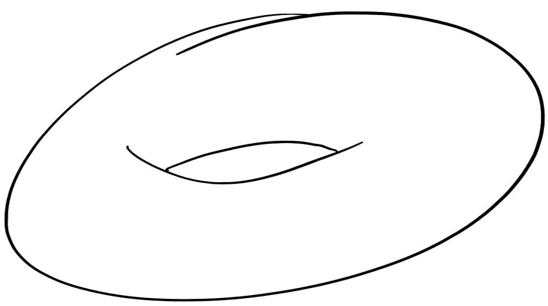
The Fourier Transform is one of the most celebrated mathematical objects for its wide range of applications to mathematics, computer science and engineering. In this paper, we present a generalization of the Fourier Transform to a wider class of spaces named LCA spaces. We conclude with the famous Pontryagin Duality Theorem.

1 Introduction

The Fourier transform is one of the most useful tools for studying functions $f : \mathbb{R} \to \mathbb{C}$. It has a wide range of applications, ranging from probability theory, to quantum mechanics to computer science! From a mathematical perspective, Fourier transform has an extremely wide impact. It has finds application in many subfields, particularly analysis, but number theory and group theory as well.

One of the key properties of the Fourier Transform is that is is very symmetrical in nature and behaves nicely under certain operations. However, as we will see, it might be better to think of the space on which the Fourier Transform acts on as the source of the symmetry, and that the Fourier Transform was built in such a way as to respect the symmetries of our space!

As it turns out, the right properties that a space should have to be able to define the Fourier Transform over it motivate the definition of a locally compact abelian group, more briefly known as LCA Groups. With this, we can generalize what spaces we define the Fourier Transform to new spaces such as a torus, which is the mathematical name for the surface of a bagel.



The paper first introduces basic notions from Group Theory in section 2. We introduce just enough so that the theorems in the later sections make sense. Similarly, we also give a light treatment of topology in section 3 so that we may understand the definition of a locally compact abelian group. For our purposes, the local compactness conditions of our spaces simply guarantee the existence of the Haar Measure, which is crucial to defining integrals over these spaces which we explain in section 4.

After setting up the background, we give a rather ad hoc treatment of Banach Algebras in section 5 which is necessary for one of the theorems we state in section 6. This section introduces the notion of a Dual Group and sketches the proof of proposition 6.10 which states that the Dual Group is an LCA group as well. This follows from theorem 6.13, which is where section 5 comes in.

The paper ends with the celebrated Pontryagin Duality theorem 7.1. This theorem asserts that a group is topologically and algebraically isomorphic to the Dual of its Dual Group. This theorem has useful corollaries such as a generalization of the Fourier Inversion Theorem 7.4 on these spaces, which we briefly mention. Other theorems that follow from Pontryagin Duality, although we don't discuss them here, include Plancherel's Theorem. The main reference for these sections was [Kat04], and it contains detailed proofs for most of the later sections.

2 Group Theory and Symmetry

One of the most important features about the Fourier Transform that we have seen is that it carries with it a lot of symmetries. It turns out that a lot of the symmetries that we have studied might actually be better understood as symmetries of the spaces over which the functions are defined!

Whenever we want to deal with symmetries, it is reasonable to think of Abstract Algebra, and particularly Group Theory. This field provides powerful tools which allow us to study these more systematically and to tell whether two symmetries are "the same," but presented differently.

Definition 2.1 (Group)**.** A **Group** is a set G with a product $* : G \times G \to G$ satisfying the following conditions:

(1) Associativity: For any $g, h, k \in G$, we have $(g * h) * k = g * (h * k)$.

(2) Identity: There is an element $1 \in G$ such that $1 * g = g * 1 = g$ for all $g \in G$

(3) Inverse: For any $g \in G$, there exists some $g^{-1} \in G$ so that $g * g^{-1} = g^{-1} * g = 1$.

◇

Example (Real Line)**.** The set of real numbers $\mathbb{R}$ are a group under addition. Indeed, addition is associative, it has the identity 0, and the negative of a number is its inverse.

It can be similarly shown that the integers under addition are a group under addition as with the real numbers. This is a group contained in a larger group which is what we call a *subgroup.*

Probably the most important example for us is the circle, which carries a natural group structure corresponding to rotations.

Example (Circle)**.** We can identify the circle with the complex numbers of norm 1. This is a group under multiplication. It also matches out with our description of rotations.

Both of the examples we gave satisfy something stronger than the group axioms. In particular, the group operation is commutative in both cases.

Remark 2.2. A group G is called **Abelian** if it is also commutative, that is, $g * h = h * g$ for all $g, h \in G$. All the groups introduced so far are commutative. ○

Remark 2.3. Let $g_0 \in G$. Left multiplication by g_0 is a map $G \to G$ given by $G \ni g \mapsto g_0 g \in G$. This is in fact a bijection. ○

For our purposes, all the groups we will study are abelian groups. In particular, we work with *continuous* abelian groups, which are groups on which the operations of inverse and composition are continuous. Note that continuity doesn't really make sense for a group yet! It turns out that assigning a group a *topology* fixes this issue, but this is something we will discuss in the next section.

Whenever we have some class of objects, one of the things mathematicians like to do is to have maps between them that somehow preserve the structure of the space. These have names that usually end in "-morphism". We begin with the algebraic morphism, which is called a homomorphism:

Definition 2.4 (Homomorphism)**.** Given two groups G_1 and G_2, a **homomorphism** $\varphi : G_1 \to G_2$ is a function such that for any $g, h \in G_1$,

$$\varphi(gh) = \varphi(g)\varphi(h)$$

so it respects the group structure.

◇

Homomorphisms between groups can be used to find structural similarities between different groups. There exists a stronger notion which tells us that two groups are *algebraically equivalent* which is called an isomorphism.

Definition 2.5 (Isomorphism)**.** An **isomorphism** is a homomorphism that is also bijective. We say that two groups G_1, G_2 are **isomorphic** if an isomorphism $\varphi : G_1 \to G_2$ exists. ◇

The reason why this defines a notion of *equivalence* in algebra is because this gives an equivalence relation on the set of groups.

An example of two groups that are isomorphic are the interval $[0, 2\pi)$ under addition modulo 2π, and the set of complex numbers of size 1 under multiplication. Because the second group is a circle in the complex plane on which each point can be identified by an angle between 0 and 2π, the relationship between the two spaces may be somewhat intuitive, but what abstract algebra has to say about these two spaces is that corresponding elements relate to each other in the same way. This is what is gained by requiring the map φ to be a homomorphism and not just a bijection.

If we visually represent each point on the circle as a rotation, then our isomorphism tells us that adding these rotations together is equivalent to adding angles together. An isomorphism essentially guarantees that algebra works the same across groups, meaning that when studying their algebraic properties, we can view them as the same. We now pivot to topology, which gives us additional structure on a set, and as a result, another notion of equivalence.

3 Topology of a Space

Topology is a branch of mathematics concerned with properties of a space that are invariant under continuous deformation. Topology is how we develop notions of closeness without explicitly using a metric. Examples of such properties that one may have been exposed to in a standard real analysis course are compactness, connectedness, and whether or not a set contains holes. Lengths and angles will not be of importance outside of their ability to establish the topological properties of a space. We will begin by defining a topology, proceed to some topological notions necessary for the development of the theory, and present some key theorems, whose proofs we will largely omit.

Definition 3.1 (Topology)**.** Given a set X, a **topology** is defined as a collection τ of subsets of X that satisfies the following properties:

(1) Both X and $\emptyset$ are in τ.

(2) The union of any collection of sets in τ, finite, countable, or uncountable, is in τ.

(3) The intersection of any finite collection of sets in τ is a member of τ.

X equipped with a topology τ is called a **topological space**. ◇

Remark 3.2. This definition is quite abstract, and may be difficult to build intuition with. As we will see again later, the key take away is this: the *topology of a space is the collection of open sets.* ○

There are two topologies that can be assigned to any non-empty set X to make it a topological space. The first is the trivial topology, which is when $\tau = \{X, \emptyset\}$. The second is the discrete topology, which is when τ contains every subset of X. We leave it to the reader to verify that these are topologies for any non-empty set X. Although trivial to construct, the set of integers equipped with the discrete topology will be an object of significant interest when we begin our study of Pontraygin duality.

Definition 3.3 (Open and Closed Sets)**.** An **open set** is a set that is a member of τ. A set is **closed** if its complement is open.

◇

If your first exposure to open and closed sets is from real analysis, the definition of open and closed sets that you are likely used to is centered around interiors and limit points. In the unit on point-set topology, distances were very much a part of defining an open set. Specifically, a set $U \subset \mathbb{R}$ was defined as open if, for every point in the set, there was an open interval containing the point that fit inside the of U. We will re-derive this definition when we introduce the metric topology on $\mathbb{R}$.

First, recall the definition of a metric.

Definition 3.4 (Metric)**.** Given a set X, a **metric** is a function d that assigns a non-negative real number $d(p, q)$ to any $p, q \in X$ in a manner that satisfies the following properties:

(1) Symmetry: For any $p, q \in X$, $d(p, q) = d(q, p)$.

(2) Definiteness: For any $p, q \in X$, $d(p, q) = 0 \iff p = q$.

(3) Triangle Inequality For any $p, q, r \in X$, $d(p, r) \leq d(p, q) + d(q, r)$.

◇

In our usual analysis classes, we usually define our open sets as **open balls**, that is, given some $x \in X$ and $\epsilon > 0$, we define $B_\epsilon(x) := \{y \in X \ : \ d(x, y) < \epsilon\}$. Morally speaking, a good topology on $\mathbb{R}$ should be such that these sets are open sets. We want a method to get such a topology given a collection of open sets. We consider the **smallest** topology containing this collection.

Proposition 3.5. *Given an arbitrary family of topologies on a set X, the intersection of all of these gives a new topology on X. In other words, given a set X with a family of topologies $\{\tau_i\}_{i\in I}$, we obtain a new topology*

$$\tau = \{U \subset X \ : \ U \in \tau_I \text{ for all } i \in I\}.$$

This means that it makes sense to speak of a ***smallest*** *topology.*

Proof. Since these are topologies, both X itself and $\emptyset$ must be contained in each topology. Now take an arbitrary collection $\{U\}_{i\in I}$ of open sets contained in each of the families. Since each is a topology, the union $\cup_{i\in I} U_i$ is in each of the families so it is in the intersection of all of these. Similarly, if we look at a finite collection of open sets in each of these topologies, then the intersection of these is also in each of the topologies.

Then we have shown that this is a topology. ■

We can now define the standard topology on $\mathbb{R}$ as the **smallest topology** so that the **open balls** are open. The metric topology is defined as the topology generated by the open intervals since open intervals and open balls are the same on $\mathbb{R}$.

Remark 3.6. The topology just described is what we call the **Standard Topology** of $\mathbb{R}$. Whenever we refer to $\mathbb{R}$ as a topological space, we are assuming that it comes with this topology. ○

The final example of a topology we will introduce is the subspace topology. This will give us a natural topology for the torus, which will be extremely important in later sections.

Definition 3.7 (Subspace Topology)**.** Let X be a topological space. Given any subset $Y \subset X$, we obtain the **subspace topology** on Y by restricting the open sets of X to Y. That is,

$$\tau_Y := \{V \subset Y \ : V = U \cap Y \text{ for some open } U \in \tau_X\}$$

is a topology on Y. We leave to the reader to check that this is in fact a topology. ◇

Consider the torus T_2, which you might recognize better as the surface of a donut (or the image at the front of the paper.) We can easily embed this into $\mathbb{R}^3$ and give it the subspace topology. This means that the open sets are obtained by intersecting the torus with open balls in $\mathbb{R}^3$.

We now reacquaint the reader with some familiar concepts from analysis, but expressed in the language of topology.

Definition 3.8 (Open Cover)**.** Given a topological space X and a subset $Y \subset X$, we say a collection of open sets $\{U\}_{i \in I}$ is an **open-cover** if any $y \in Y$ belongs to some open set $y \in U_i$. A **sub-cover** is a subcollection of a cover. ◇

Definition 3.9 (Compactness)**.** A **compact** set K is one such that for every open cover of K, there exists a finite sub-cover. ◇

This seems unintuitive, but the following theorem may help make this feel more concrete:

Theorem 3.10 (Heine-Borel)**.** *Under the standard topology on* $\mathbb{R}$*,* $K \subset \mathbb{R}$ *is compact if and only if it is closed and bounded.*

Proof. We refer the reader to [Rud76]. ■

Definition 3.11 (Continuity)**.** Let X, Y be two topological spaces. A function $f : X \to Y$ is **continuous** if for every open set $U \in Y$, the preimage of U in f, denoted $f^{-1}(U)$, is open in X. ◇

We now introduce the notion of a homeomorphism, which, similarly to an isomorphism, signifies a type of equivalence between two spaces, this time in terms of their topological properties.

Definition 3.12 (Homeomorphism)**.** Given two topological spaces X and Y, a *homeomorphism* $f : X \to Y$ is a continuous function with a continuous inverse. ◇

Given a homeomorphism $f : X \to Y$ with inverse g, we must have that for an open set $U \subset X$, we must have that the pre-image $g^{-1}(U)$ is open, but since this is the inverse of f, $g{-}1(U) = f(U)$, so this maps open sets to open sets and vice versa.

Herein lies our notion of equivalence: This gives a bijection of open sets in X to open sets in Y. Topologies are defined completely by open sets, so from a topological perspective, these two sets are equivalent. This is analogous to what the isomorphism was for algebra.

We now introduce the objects on which we can develop Fourier transform in its full generality. We begin with the definition of a topological group, a logical synthesis of the preceding two sections.

Definition 3.13 (Topological Group)**.** A **topological group** G is a topological space with a group structure such that the following are continuous:

(1) **Multiplication** $* : G \times G \to G$ given by $(x, y) \mapsto x * y$

(2) **Inversion** $-1 : G \to G$ given by $x \mapsto x^{-1}$

◇

Remark 3.14. Notice that above we need a topology for $G \times G$ to be able to say that multiplication is continuous. Let's give it a topology to finish this definition. ○

Definition 3.15 (Product Topology)**.** Given two topological spaces (X, τ_X) and (Y, τ_Y), we define the product topology as the following:

$$\tau_{X \times Y} := \{U \times V \subset X \times Y \; : \; U \in \tau_X, V \in \tau_Y\}.$$

◇

Example. $(\mathbb{R}, +)$ equipped with the standard topology, $(\mathbb{Z}, +)$ equipped with the discrete topology, and the circle equipped with the subspace topology when identified with the unit complex numbers are all topological groups.

It turns out that to define our theory, we want to enforce some nice conditions on our space.

Definition 3.16 (Local Compactness)**.** A topological group G is **locally compact** if it possesses two qualities:

(1) For every point $x \in G$, there exists an open set U and a compact set K such that $x \in U \subset K$.

(2) G is Hausdorff, a topological property meaning for any two points $x, y \in G$, there are open sets $U, V \subset G$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

◇

Remark 3.17. A group G is an LCA (Locally Compact Abelian) group if G is locally compact and Abelian. ○

Remark 3.18. All of the groups listed above are also LCA groups and will be the protagonists of this paper. ○

The conditions listed above are technical, but far from arbitrary. Topological groups that are locally compact carry Haar measures, measures that are translation-invariant.

4 Haar Measure on LCA Groups

We've seen in class that when f is a function over $\mathbb{R}$, the Fourier transform of f, when it exists, is a function defined on the very same space. As we extend our discussion of Fourier transform to different spaces, we will observe that this is rarely the case, and we will need a definition of integration that is more general. As has been alluded to in class, the Riemann integral is insufficient. This section and the next can be skipped if the reader is simply interested in the main results.

We will briefly introduce the notion of a measure, define the Lebesgue measure on $\mathbb{R}$, and use it as our primary example for integrating with respect to a measure. This is defined in section A, but we refer the reader to more classic texts on the matter, such as [Dur19].

Definition 4.1 (σ-Algebra)**.** Let X be a set. A **σ-algebra** is a collection $\mathcal{F}$ of subsets of X that is closed under countable unions, countable intersections, and complementation. A set X equipped with a σ-algebra is called a *measurable space.* ◇

We introduce the notion of a σ-algebra because not all sets can be given a measure. To remedy this, we restrict our attention to a smaller collection of sets, typically by introducing sets of particular interest,and then adding the necessary sets to make $\mathcal{F}$ a valid σ-algebra. We refer to the sets in $\mathcal{F}$ as *measurable sets.* The most common σ-algebra is the Borel σ-algebra, generated by taking countable unions, countable intersections, and complements of open sets. We can now fully define the notion of measure.

Definition 4.2 (Measure)**.** Let X be a set, and $\mathcal{F}$ be a σ-algebra. A **measure** $\mu : \mathcal{F} \to \mathbb{R}$ is a function that satisfies the following properties:

(1) Non-Negativity: For every measurable set E, $\mu(E) \geq 0$.

(2) Countable Additivity: Given a collection $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint measurable sets, $\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k)$.

(3) Null Empty Set $\mu(\emptyset) = 0$.

A triplet $(X, \mathcal{F}, \mu)$ is referred to as a *measure space.* ◇

Example. Let $X = \mathbb{Z}$ and let $\mathcal{F} = 2^{\mathbb{Z}}$, the power set of Z. The *counting measure* μ is defined as follows:

$$\mu(E) = \begin{cases} |E| \text{ if } |E| \text{ is finite} \\ \infty \text{ otherwise} \end{cases}$$

Definition 4.3 (Sigma-Finite)**.** Given a measurable space $(\Omega, \mathcal{F}, \mu)$, we say it is **sigma-finite** if there exists a countable family $\{A_n\}_{n=1}^{\infty} \in \mathcal{F}$ such that $\Omega = \cup_{n \in \mathbb{N}} A_n$ and each $\mu(A_n) < \infty$. ◇

We are almost ready to define integration! We first need to introduce the notion of a measurable function.

Definition 4.4 (Measurable Function)**.** Let $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ be measurable spaces, a function $f : X \to Y$ is called **measurable** if the pre-image of a measurable set is measurable. ◇

Compare this with the definition of a continuous function in Definition 3.11. These are very similar notions and indeed, we want these to be related. Recall that the intersection of a family of topologies is a topology, there is an analogue for sigma-algebras.

Remark 4.5. Arbitrary intersection of sigma-algebras is a sigma-algebra. ○

What this means for us is that it makes sense to speak of a *smallest* sigma-algebra.

The following example will be very important to us:

Definition 4.6 (Borel Sigma Algebra)**.** Given a topological space (X, τ), we define the **Borel Sigma Algebra** $\mathcal{B}$ as the smallest sigma-algebra containing the open sets, i.e. $\tau \subset \mathcal{B}$. ◇

Now that we have this, let's once more think about the interplay between continuous and measurable functions. It turns out that we have the following;

Remark 4.7. Given two topological spaces (X, τ) and (Y, τ'), any continuous function between these is measurable if we equip them with the Borel sigma-algebra, ◦

It turns out, this is *almost* the sigma-algebra we want on $\mathbb{R}$. The real sigma-algebra we want is obtained from the following notion.

Definition 4.8 (Complete Measure Space)**.** Given a measure space $(\Omega, \mathcal{F}, \mu)$, we say it is complete if for all $N \in \mathcal{F}$ of measure 0, we have $S \subset N \implies S \in \mathcal{F}$ with measure 0 too. ⋄

Theorem 4.9 (Uniqueness of Completion)**.** *Any measure space has a unique completion.*

Proof. We refer the reader to [Dur19]. ■

Now, in the case of $\mathbb{R}$ with the Borel sigma-algebra, there is a unique measure so that the measure of a set of the form (a, b) has measure $|b - a|$. This is called the Borel measure. The Lebesgue measure is the completion of this, so that we extend this to the null-sets. The extension of the Riemann Integral into the Lebesgue Integral is in some way the same idea.

Recall that our protagonists are **locally compact groups**. Our focus from a theoretical standpoint will be to find a measure that guarantees our ability to do Fourier transform. Some of the requirements on that measure are somewhat technical, but the most important for our purposes is translation-invariance, defined as follows:

Definition 4.10 (Left-Invariant)**.** We refer to a measure μ on $(G, \mathcal{E})$ as **translation-invariant** if for any measurable set E and element $g \in G$, $\mu(gE) = \mu(E)$, where $gE = \{gx | x \in E\}$. If we instead multiply on the right, we call this **right-invariant**. ⋄

The importance of local compactness in the development of our theory is found in the following theorem:

Theorem 4.11. *Every locally compact abelian group has what's known as a* ***Haar measure****, which is a left-invariant and right-invariant measure on our group. It is unique up to a multiplicative constant.*

Proof. We refer the reader to [Gle10] for a more precise treatment, but the idea is to build an index $(A : U)$, defined as the minimum number of translated copies of U needed to cover A. We can compare this to $(K : U)$, with K some compact set in X. As U gets smaller, this converges. The quantity to which this converges is our Haar measure. ■

Remark 4.12. The Haar measure exists for groups that aren't neccesarily abelian, but in these cases, we are not guaranteed right-invariance. ◦

Remark 4.13. On $\mathbb{R}$ and S, the Haar measure is the Lebesgue measure. On $\mathbb{Z}$, the Haar measure is the counting measure.

◦

Now that we have finally equipped our spaces with the Haar measure, we can define $L^1(G)$ as the space of integrable and measurable functions with respect to this measure. One can check that this is a vector space and we will study this deeply in the next section.

5 Banach Algebras

We defined the space of integrable functions $L^1(G)$ for topological groups G using the Haar Measure. The reason why we refer to it as a space is because it is actually a vector space over $\mathbb{C}$. Indeed, it is closed under addition and scalar multiplication.

Although $L^1(G)$ is a vector space, one should really be careful about how we think of this.

Warning 5.1. $L^1(G)$ does not need to be a finite dimensional vector space.

Proof. We show the case when $G = (\mathbb{R}, +)$ only. Consider the family $\{f_n\}_{n\in\mathbb{N}}$ given by $f_n = 1_{[n,n+1)}$, so it is 1 if $x \in [n, n+1)$ and 0 otherwise. It has infinitely many elements and they are all linearly independent. Therefore, $L^1(\mathbb{R})$ cannot be finite dimensional. ■

Still, even for an infinite dimensional space, $L^1(G)$ has a nice structure. First, recall the following definition:

Definition 5.2 (Completeness)**.** We say a metric space (X, d) is **complete** if every Cauchy sequence converges, ◇

It will turn out that $L^1(G)$ comes naturally with a norm under which it is complete. These type of spaces are widely studied and thus have their own name:

Definition 5.3 (Banach Space)**.** A **Banach Space** is a complete, normed vector space. ◇

Theorem 5.4. *For the case $L^1(G)$, the norm is given by*

$$\|f\|_1 := \int |f| d\mu$$

which is the integral with respect to its Haar measure. The space is complete with respect to this norm.

Proof. We refer the reader to [Ste07] for a proof in the case of the real numbers (Riesz-Fischer.) ■

Remark 5.5. All integrals will be with respect to the Haar measure. We will write dx at the end of the integral to specify with respect to what variable we are integrating. ○

We can give this space even more structure. Recall the convolution operation for the case over the real line $\mathbb{R}$. We can generalize this for $L^1(G)$.

Definition 5.6 (Convolution)**.** The **convolution** of two functions $f, g \in L^1(G)$ is defined by

$$(f * g)(x) := \int_G f(y) g(xy^{-1}) dy$$

whenever this integral exists. ◇

It turns out that convolution gives our space even more structure. To state this, we first have to give some more definitions.

Definition 5.7 (Algebra)**.** An **Algebra** $\mathcal{A}$ over $\mathbb{C}$ is a complex vector space with an associative bilinear product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$. ◇

Remark 5.8. An algebra is called *commutative* if the product is commutative. ○

As always, we want to define the notion of a homomorphism between algebras.

Definition 5.9 (Algebra Homomorphism)**.** An algebra homomorphism between two algebras A and B is a $\mathbb{C}$-linear[1] map $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$. ◇

If the set up so far has been clear, you might already be thinking that what we'll show is that $L^1(G)$ is an algebra. In that case, you would be right. However, let's give just one more definition.

Definition 5.10 (Banach Algebra)**.** A **Banach Algebra** is an algebra $\mathcal{A}$ equipped with a norm $\|\cdot\|$ under which $\mathcal{A}$ is complete. Further, the norm should be *submultiplicative*, so it satisfies $\|ab\| \leq \|a\|\|b\|$. ◇

In other words, a Banach algebra is a Banach Space that is also an algebra, under which the norm is submultiplicative. Now we can fully state our theorem:

Theorem 5.11. *For a locally compact group G, the space $L^1(G)$ of integrable functions with the usual norm $\|\cdot\|_1$ is a Banach Algebra under* ***convolution****.*

Proof. 5.4 asserts that $L^1(G)$ is complete. We will only show that the convolution indeed gives an $L^1(G)$ function and is submultiplicative with respect to the norm. Associativity is left to the reader (for the full proof, we refer the reader to [Kat04].)

The proof of this uses non-trivial facts about product measures and Fubini's theorem, which we don't discuss rigorously[2]. The harder part is really showing that $f * g$ is an $L^1(G)$ function, so we will simply take it as given. Assuming these, we see that

$$\|f * g\|_1 = \int_G \left| \int_G f(y)g(xy^{-1})dy \right| dx \leq \int_G \int_G |f(y)g(xy^{-1})|dydx$$

and by Fubini's theorem, we can switch these, so

$$= \int_G \int_G |f(y)g(xy^{-1})|dxdy = \int_G |f(y)| \left(\int_G |g(xy^{-1})|dx \right) dy$$

and since the Haar measure is translation invariant, $\|g\|_1 = \int_G |g(x)|dx = \int_G |g(xy^{-1})|dx$ and so we have

$$= \|g\|_1 \int_G |f(y)|dy = \|f\|_1 \|g\|_1$$

which completes the proof. ■

[1] $\mathbb{C}$-linear meaning the map is linear and you can pull out complex constants ($\varphi(\lambda x) = \lambda\varphi(x)$ for $\lambda \in \mathbb{C}$.)

[2] We refer those seeking a rigorous treatment of Fubini's theorem to [Dur19].

Then we have shown that indeed, $L^1(G)$ is a Banach Algebra under convolution. We are interested in $L^1(G)$ for a locally compact abelian group, so it is natural to ask whether $L^1(G)$ is commutative in this case. It turns out that this is the case.

Proposition 5.12. *$L^1(G)$ is a commutative Banach Algebra under convolution if and only if G is itself abelian.*

Proof. We refer the reader to [Kat04]. ∎

Now that we have studied the properties of $L^1(G)$, one can consider things *acting* on this space. In particular, we will consider the space of bounded linear functionals acting on this space, which is well defined for any Banach space.

Definition 5.13. Given a Banach space X, we define the **dual space** X' to be the set of all bounded linear functionals $T : X \to \mathbb{C}$. ◇

Proposition 5.14. *The Dual Space is itself a Banach space under a natural norm. Let $S(X) := \{x \in X \ : \ \|x\| = 1\}$ which we call the unit sphere. Then the norm is given by the following:*

$$\|T\| := \sup_{s \in S} \|Ts\|$$

Proof. We refer the reader to [Ste07]. ∎

This space is defined for any Banach space, but our protagonist in this section is $L^1(G)$, which is a *commutative Banach algebra.* This carries even more structure now, since now we can multiply by $\mathbb{C}$ and it has a commutative multiplication. Then the following definition makes sense:

Definition 5.15 (Structure Space)**.** The **structure space** $\Delta_{\mathcal{A}}$ of a commutative Banach algebra $\mathcal{A}$ is defined as the set of non-zero continuous homomorphisms $\mathcal{A} \to \mathbb{C}$. ◇

Notice that this is the same as a functional, since we need $\varphi(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \varphi(x_1) + \lambda_2 \varphi(x_2)$. However, now we can multiply elements as it is an algebra, so we could possibly want $\varphi(xy) = \varphi(x)\varphi(y)$. Since complex numbers are commutative, we have $\varphi(xy) = \varphi(x)\varphi(y) = \varphi(y)\varphi(x) = \varphi(yx)$ so for this to be well-defined, we require that our algebra be commutative. Then this is really just the same as a **multiplicative linear functional.**

Later on, this space will come up again, but when that time comes, we will mainly be interested in its topology.

Theorem 5.16. *There is a natural topology of the structure space $\Delta_{\mathcal{A}}$ under which it is a locally compact, Hausdorff space.*

We won't prove this in its entirety, but we spend the rest of this section building up towards the idea of the proof. Before this, we need more background.

Proposition 5.17. *For a commutative algebra $\mathcal{A}$, every algebra homomorphism $m : \mathcal{A} \to \mathbb{C}$ is a bounded linear functional, that is, $\Delta_{\mathcal{A}} \subset \mathcal{A}'$. Further, $\|m\| \leq 1$ for all $m \in \Delta_{\mathcal{A}}$.*

Proof. It follows from the definition that this is a linear functional, so we just need to show it is bounded. Recall that $S(X)$ (defined in 5.14) is the unit sphere in X. Assume that there exists some $s \in S(\mathcal{A})$ such that $\|m(s)\| > 1$. Let $x = s/\|m(s)\|$ so that $\|m(x)\| = 1$ and $\|x\| < 1$. Without loss of generality, we can multiply x by some $e^{i\theta}$ to make $m(x) = 1$. Then let $y = x + x^2 + ...$ which converges. Then we have that $x + yx = y$ and so

$$1 + m(y) = m(x) + m(y)m(x) = m(x + yx) = m(y)$$

which is a contradiction! So for all $s \in S$, we must have $\|m(s)\| \leq 1$. ■

So we see that $\Delta_{\mathcal{A}}$ is a subset of the unit ball $B := \{T \in X' \ : \ \|T\| \leq 1\}$. What we will now do is show that there is a natural topology on X' which makes the unit ball into a locally compact Hausdorff space. This then would prove that $\Delta_{\mathcal{A}}$ is also locally compact and Hausdorff under the *subspace topology*.

Definition 5.18 (Weak-* Topology)**.** The weak-* topology on X' is the smallest topology under which bounded linear functionals are continuous. ◇

Remark 5.19. Note that this is indeed well-defined as the intersection of a family of topologies is a topology, so it really makes sense to talk about a "smallest" topology. ○

The following theorem is quite complicated and we will not prove it; in fact, we have not built up the technology to prove it in this paper. We simply state it:

Theorem 5.20 (Banach-Alaoglu)**.** *Let V be a normed, complex vector space*[3]*. Then*

$$B := \{T \in V' \ : \ \|T\| \leq 1\}$$

is locally compact and Hausdorff under the weak- topology.*

Proof. We refer the reader to [Kat04]. ■

Now, we could give a proof for Theorem 5.16 using the above theorem, but we won't do this. The proof is the following: You can show that the natural inclusion of $\Delta_{\mathcal{A}} \hookrightarrow B$ is in fact a topological embedding and then the theorem immediately follows. Showing this is a bit technical, so we will admit it as true. The reader may refer to [Kat04], which outlines the proof as we have and proves it in full.

6 Dual Groups and the Fourier Transform

We can finally put together all that we've been working for and define the Fourier Transform on LCA groups.

[3]Note that we do not require this to be Banach.

Definition 6.1 (Dual Groups)**.** We define the **Dual Group** $\widehat{G}$ of an LCA group G to be the group of continuous homomorphism from G to the circle, i.e. $\widehat{G} := \mathrm{Hom}(G, S^1)$. The elements of $\widehat{G}$ are called the **characters** (χ) of G. ◇

Remark 6.2. With how the theory has been set up so far, the fact that we are looking at homomorphism to S^1 might seem like an arbitrary choice. However, when you consider $S^1 \subset \mathbb{C}$ as the unit complex numbers, it turns out that from the viewpoint of **Representation Theory**, this is actually a quite natural choice. Even the name *characters* is a convention from Representation Theory. ○

Remark 6.3. The Dual Group $\widehat{G}$ is a group under pointwise multiplication: $\chi_1 \times \chi_2 \mapsto \chi_1\chi_2$. The identity homomorphism is the trivial homomorphism: $\chi(x) = 1$ for all x and the inverse is given by $\chi^{-1} = \overline{\chi}$ since $\chi(x)\overline{\chi(x)} = 1$ for all x. Associativity follows from multiplication of the complex numbers. Notice that on top of this, it is also Abelian since mulitplication over the complex numbers is commutative. ○

These groups turn out to be the main characters for the Fourier Transform. In particular, the Fourier Transform will be a map from functions in $L^1(G)$ to functions in $L^1(\widehat{G})$.

Definition 6.4 (Fourier Transform)**.** The **Fourier Transform** of a function $f \in L^1(G)$ is a function $\widehat{f} : \widehat{G} \to \mathbb{C}$ defined on $\widehat{G}$ by

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx$$

◇

Before explaining too much about this, let's recover the cases that we already know from this.

Proposition 6.5. *The Dual Group of* $\mathbb{R}$ *is* $\mathbb{R}$ *itself.*

To this end, we will use the following lemma which relies on ideas from the study of Lie Groups.

Lemma 6.6. *Any continuous homomorphism* $\mathbb{R} \to S$ *is differentiable.*

Proof. We refer the reader to [Lee00]. ■

Now this gives for a nice proof of Proposition 6.5.

Proof. Consider $\varphi : \mathbb{R} \to S$ a continuous homomorphism. By 6.6, it is a *differentiable* homomorphism. Notice that

$$\varphi'(x) = \lim_{h\to 0} \frac{\varphi(x+h) - \varphi(x)}{h} = \lim_{h\to 0} \frac{\varphi(x)\varphi(h) - \varphi(x)\varphi(0)}{h}$$

$$= \varphi(x) \lim_{h\to 0} \frac{\varphi(h) - \varphi(0)}{h} = \varphi(x)\varphi'(0).$$

Then we have a differential equation $\varphi'(x) = \varphi(x)\varphi'(0)$ with initial data $\varphi(0) = 1$, and so the solutions to this equation are of the form $\varphi(x) = e^{isx}$ where $is = \varphi'(0)$. Thus we have shown that $\widehat{\mathbb{R}} \ni \varphi \mapsto \varphi'(0) \in \mathbb{R}$ gives a bijection between the reals and the dual group of $\mathbb{R}$. ∎

As a general rule of thumb, whenever a theorem in mathematics claims two different objects are isomorphic, the part of the theorem that one should pay attention to is the isomorphism itself. Just knowing the statement of 6.5, we wouldn't really be able to compute the Fourier Transform, since we wouldn't know how the elements of the Dual Group act on the Group itself. However, we found a natural isomorphism between them in the proof of our theorem, and we can use this to compute the Fourier Transform over the real numbers.

Corollary 6.7. *The Fourier Transform on the real line is the operator taking $f \in L^1(\mathbb{R})$ to a complex-valued function $\widehat{f}$ over the real line by*

$$\widehat{f}(s) := \int_{\mathbb{R}} f(x)e^{-isx}dx$$

Now let's recover another example we are familiar, namely, the *Fourier Series.*

Proposition 6.8. *The Dual Group of S is $\mathbb{Z}$.*

Proof. Say $\varphi : S \to S$ is a continuous homomorphism. Then the map $\psi : \mathbb{R} \to S$ given by $\psi(x) = e^{ix}$ gives a continuous homomorphisms $\mathbb{R} \to S$. Since the composition of continuous homomorphisms is a continuous homomorphism, then this is a new continuous homomorphism $\mathbb{R} \to S$, so it is given by $\xi : x \mapsto e^{isx}$ for some $s \in \mathbb{R}$. Since we need that $\xi(2\pi) = 1$, then we must have that s is an integer. Thus the dual group of S is given by the integers. ∎

Corollary 6.9. *The Fourier Transform on the circle is the operator taking $f \in L^1(S^1)$ to a complex-valued function $\widehat{f}$ over the integers ($\mathbb{Z}$) by*

$$\widehat{f}(n) := \int_{S^1} f(x)e^{-inx}dx$$

Notice that there is a natural bijection between functions on the circle and functions on $[0, 2\pi]$ with the endpoints identified, so we can rewrite this as

$$\widehat{f}(n) = \int_0^{2\pi} f(x)e^{-inx}dx$$

which is the Fourier Coefficient for $2\pi-$periodic functions!

Notice that all we've said so far is that this is a *function* and we have said nothing about the continuity of the Fourier Transform. This might've been easier to miss for the integers since we don't usually care about continuity for functions over the integers, but this is something we are interested in knowing for the real numbers. Those familiar with the Fourier Transform over the real line might remember that with the usual topology of $\mathbb{R}$, the Fourier Tranform of an integrable function is continuous, so one might imagine that there is a natural topology for the Dual Groups. In fact, something stronger will be true:

Proposition 6.10. *The Dual Group of an LCA group is an LCA group.*

We won't prove this completely, but we will define the topology under which it is such a group and sketch the proof. First, let's give the topology:

Definition 6.11 (Compact-Open Topology)**.** Given a topological space X, denote the vector space of complex-valued continuous functions on X by $C(X)$. Then given a compact set $K \subset X$ and an open set $U \subset \mathbb{C}$, we define

$$L(K, U) := \{f \in C(X) \mid f(K) \subset U\}$$

The topology generated by these sets is called the **compact-open topology.** ◇

This definition might be scary and alien, but it is actually much more familiar to us. We won't prove the following remark.

Remark 6.12. The compact-open topology on the following spaces agrees with the following topologies:

(1) On $\mathbb{R}$, it agrees with the usual topology.

(2) On S, it agrees with the usual topology.

(3) On $\mathbb{Z}$, it agrees with the discrete topology.

○

We know these are locally compact and Hausdorff, so this doesn't contradict our claim. However, let's see why this should be so.

Theorem 6.13. *Given $\chi \in \widehat{G}$, we can define the following functional on $L^1(G)$:*

$$d_\chi(f) = \widehat{f}(\chi)$$

*and this is a **homeomorphism** onto $\Delta_{L^1(G)}$.*

So we have changed our perspective on the Fourier Transform: instead of thinking of it as mapping a function $f \in L^1(G)$ into a new function $\widehat{f} \in L^1(\widehat{G})$, we think of an element $\chi \in \widehat{G}$ as a functional acting on functions $f \in L^1(G)$ by evaluating the Fourier Transform on it!

We won't show that this is an isomorphism, but hopefully one can at least understand why this map between the two spaces is quite natural. Now let's say one more thing about the Fourier Tranform before moving on:

Theorem 6.14 (Convolution Theorem)**.** *Given $f, g \in L^1(G)$, $\widehat{f * g} = \widehat{f}\widehat{g}$.*

Proof. We again rely on Fubini's theorem.

$$\widehat{f * g}(\chi) = \int_G (f*g)(x)\overline{\chi}(x)dx = \int_G \left(\int_G f(y)g(xy^{-1})dy \right) \overline{\chi}(x)dx = \int_G \int_G f(y)g(xy^{-1})\overline{\chi}(x)dydx$$

$$= \int_G \int_G f(y) g(xy^{-1}) \overline{\chi}(x) dx dy = \int_G f(y) \left(\int_G g(xy^{-1}) \overline{\chi}(x) dx \right) dy$$

Consider the change of variables $x' = xy^{-1}$. As the Haar measure is translation invariant, this gives us that

$$\int_G g(xy^{-1}) \overline{\chi}(x) dx = \int_G g(x') \overline{\chi}(x') \overline{\chi(y)} dx'.$$

Then we can substitute this and so we have

$$= \left(\int_G f(y) \overline{\chi(y)} dy \right) \left(\int_G g(x) \overline{\chi(x)} dx \right) = \widehat{f}(\chi) \widehat{g}(\chi)$$

■

7 Pontryagin Duality

We finally get to the celebrated Pontryagin Duality in this section. This could be considered the main result of this paper and as a consequence, we obtain the Fourier Inversion Theorem and Plancherel's Theorem. We will exhibit the isomorphism of the statement of the theorem, but we will not prove it is so.

The big result at the end of the previous section was the fact that the dual group $\widehat{G}$ of an LCA group is also an LCA group. Then one could consider the dual of the dual group! We can write this as $\widehat{\widehat{G}}$. An interesting thought one could have is whether we could keep taking duals to create new LCA groups. It turns out that this won't quite work. Pontryagin Duality asserts the following: there exists an isomorphism $\delta : G \to \widehat{\widehat{G}}$ called the *Pontryagin Map*.

Theorem 7.1 (Pontryagin Duality)**.** *An LCA group G is isomorphic to the dual of its dual $\widehat{\widehat{G}}$.*

Proof. We refer the reader to [Kat04] for a full proof. ■

Notice that we didn't define the map. Let's think about what this might be. An element $\chi \in \widehat{G}$ is a homomorphism

$$\chi : G \ni x \mapsto \chi(x) \in S^1.$$

However, what if instead of fixing χ and letting x vary, we fix x and let χ vary? We would get the homomorphism

$$\delta_x : \widehat{G} \ni \chi \mapsto \chi(x) \in S^1.$$

then we have found a natural map $\delta : G \to \widehat{\widehat{G}}$ by $\delta(x) = \delta_x$ as above! It is still left to show that such a map is indeed an isomorphism of LCA groups, that is, that is an algebraic and topological isomorphism.

Remark 7.2. The above map is called the *Pontryagin Map* $\delta : G \to \widehat{\widehat{G}}$. It is the same as the Dirac Delta, as it acts by evaluation. ◦

A nice corollary of Pontraygin duality is that finding $\hat{G}$ gives us $\hat{\hat{G}}$ for free. For example, if we'd like to show that the dual group of S is $\mathbb{Z}$, this can be done by showing that the torus is the dual group of the integers–which turns out to be a far easier task because of the discreteness of the integers.

Corollary 7.3. *The dual group of the circle $\widehat{S}$ is isomorphic to the integers $\mathbb{Z}$. This is the statement of 6.8.*

Proof. We prove the opposite, and then our desired statement follows from Pontryagin duality. Because $\mathbb{Z}$ is generated by 1, meaning every element in $\mathbb{Z}$ can be reached by adding or subtracting 1, defining a homomorphism from the integers to the circle requires us only to decide where 1 is mapped to. Then φ is determined by the image of 1, so $\varphi(1) \in S$ fixes a homomorphism. This means that the dual group of $\mathbb{Z}$ is S as a set. If we equip the homomorphisms with the operation of multiplication, we see that if $\varphi_1 = e^{in\theta_1}$ and $\varphi_2 = e^{in\theta_2}$, then $(phi_1 * phi_2)(n) = e^in\theta_1 + \theta_2$. From this it is clear that this group of homomorphisms is isomorphic to the circle. By Pontryagin Duality, this means that the dual group of S is $\mathbb{Z}$, which completes the proof. ∎

We can now use the power of the Pontryagin Map to prove the Fourier Inversion Theorem. The technology needed to prove this rigorously is actually developed during the proof of Pontryagin Duality, so we will give incomplete proofs.

Let's think about what Fourier Inversion Theorem might be. The usual Fourier Transform is a map

$$f(x) \mapsto \widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx.$$

Now, say that we do this again, then we get a map

$$f(\chi) \mapsto \widehat{f}(\delta_x) = \int_{\widehat{G}} f(\chi)\overline{\chi(x)}d\chi$$

where we know we can write it as a function of x since $G \cong \widehat{\widehat{G}}$ by Pontryagin Duality! Then composition gives a map $f \mapsto \hat{\hat{f}}$ of a new function over G. This is almost the map we want: It turns out that what we want is for the mapping from a function on $\widehat{G}$ to one on G to have $\delta^{-1} : x \mapsto \delta_{x^{-1}}$ for things to work out nicely.

Notice that we haven't said anything about what type of functions we are using, i.e. whether $f \in L^1(G)$, or something else. To rigorously prove this, we would need to properly define these spaces and even more.

Theorem 7.4 (Fourier Inversion)**.** *The following functions over G are the same, whenever they are well-defined:*

$$f(x) = \widehat{\widehat{f}}(\delta_{x^{-1}})$$

Proof. We refer the reader to [Kat04]. ∎

One can further prove other theorems, such as Plancherel's, but to state them in their full generality, we would need more technology.

Pontryagin Duality isn't the end of the story: If we look at non-abelian cases, we get a lot of different examples of locally compact, Hausdorff spaces with a group structure. For example, $GL_n(\mathbb{C})$, $U_n(\mathbb{C})$ and others are all locally compact and Hausdorff with a group structure. These particular examples are what are known as Lie Groups, which are of great mathematical interest.

Appendices

A The Measure Theoretic Approach to Integration

The goal of this section is to be able to give a measure-theoretic definition of the integral. The ideal way to do this is to define it for increasing levels of complexity:

(1) Simple functions (i.e. piece-wise constant.)

(2) Bounded, finitely supported, measurable functions.

(3) Nonnegative, measurable functions.

(4) Measurable functions.

This is defined somewhat inductively, where the *nth* definition gives you the *n+1th* definition. The first one is the least intimidating and easiest to attack. The following definitions will be for a measurable space $(\Omega, \mathcal{F}, \mu)$ with sigma-finite measure, and remember that the characteristic function is defined as $1_A(\omega)$ is 1 if $\omega \in A$ and 0 otherwise for any $A \in \mathcal{F}$.

Definition A.1. A function f is called **simple** if the following holds:

$$f(\omega) = \sum_{i=1}^{n} a_i 1_{A_i}(\omega) \text{ such that } \mu(A_i) < \infty \text{ for all i, and } A_i \cap A_j = \emptyset \text{ for } i \neq j.$$

◇

In simpler terms, we can find a finite number of measurable sets on which the function is constant, and the function is simply the sum of these. The integral definition is quite intuitive:

Definition A.2 (Integral 1)**.** The **integral** for a *simple* function is defined as

$$\int_\Omega f d\mu = \sum_{i=1}^{n} a_i \mu(A_i) \text{ where } f(\omega) = \sum_{i=1}^{n} a_i 1_{A_i}(\omega)$$

◇

Now we want to generalize this more. The first expansion will be to consider some bounded, measurable f such that $\text{supp}(f) := \{\omega \in \Omega \ : \ f(\omega) \neq 0\}$ is contained in some $S \in \mathcal{F}$ with $\mu(S) < \infty$.

The idea is the following: we want to approximate such a function by simple functions to obtain its integral. This process might remind you a bit of the Riemann Integral, where we approached our desired integral with rectangles from a partition. We give the following definition:

Definition A.3 (Integral 2)**.** The **integral** of a *bounded, measurable* function with $\text{supp}(f) \subset S \in \mathcal{F}$ with $\mu(S) < \infty$ is defined as

$$\int f d\mu = \sup_{\alpha \text{ simple with } \text{supp}(\alpha) \subset S \text{ and } \alpha \leq f} \int \alpha d\mu$$

It can equivalently be defined for infimum with $\alpha \geq f$, and we can prove that these agree, although we won't do this here. ◇

Now we're moving up in the world. The last class of functions was quite restrictive, but the current one now lets us integrate a lot more things! Still, there are a lot of functions that we might want to integrate that don't have finite support. As an example, the Gaussian is never zero!

We now define it for non-negative measurable functions, and then going from there to measurable functions is simpler. We again won't prove why we choose non-negative, but this might give some intuition as to why we choose this: The idea is that we want to approach it by a family of functions from below, and to make sure these are always bounded, the family we approach by is $f_n(\omega) := \min(n, f(\omega))$. We now give the definition.

Definition A.4 (Integral 3)**.** Given non-negative and measurable f, we define the **integral** as

$$\int_\Omega f d\mu = \sup_{0 \leq h \leq f} \int_\Omega h d\mu \text{ for h satisfying A.3}$$

◇

We again won't give a proof, but now that we have this, we can define the integral for measurable functions.

Definition A.5 (Integral 4)**.** Given measurable f, let $f^+ := \max(f, 0)$ and $f^- := \max(-f, 0)$. Then

$$\int_\Omega f d\mu = \int_\Omega f^+ d\mu - \int_\Omega f^- d\mu$$

◇

and so we have finally defined the integral.

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