# Khovanov Homology

Carlos Andrés Alvarado Álvarez

**Abstract**

The goal is to introduce Khovanov homology and TQFTs and see how we can recover the Khovanov homology from applying some TQFT. In the process, we sketch the classification of 2-TQFTs.

## 1 Introduction

One of the fundamental ways in which mathematicians tell objects apart is by looking at invariants of said objects. Since we are interested in knot theory, then a plan of attack for us might be to find knot invariants to tell two knot diagrams apart.

Here is an idea: we can take any crossing in a knot diagram $\langle \times \rangle$ and *smooth* it so that it's not a crossing by either $\langle )( \rangle$ or $\langle \asymp \rangle$ so this gives us two new knots with one less crossing. Could we make an invariant that somehow depends recursively on these two simpler knots?

It turns out that we indeed *can* make such an invariant!

**Theorem 1.1** (Jones polynomial)**.** *We can associate to a knot diagram a knot invariant known as the* ***Jones polynomial*** *which is characterized by the following axioms:*

$$\langle \emptyset \rangle = 1; \quad \langle \bigcirc L \rangle = (q + q^{-1})\langle L \rangle; \quad \langle \times \rangle = \langle \asymp \rangle - q\langle )( \rangle.$$

*and for $n_+, n_-$ the number of positive and negative crossings respectively, we have:*

$$J(L) = \frac{(-1)^{n_-} q^{n_+ - 2n_-}}{(q + q^{-1})} \langle L \rangle$$

*as the Jones polynomial.*

*Remark* 1.2. The brackets by themselves are just called the *bracket polynomial.* We could also have not divided by $(q + q^{-1})$ and ended up with what we call the *unnormalized Jones polynomial.* ◦

These axioms give us all the computational power we need! Let's use them to compute the Jones polynomial for the trefoil.

**Example** (Jones polynomial for trefoil)**.** Consider the following diagram for the trefoil:

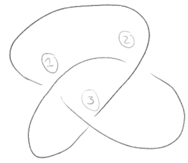

Figure 1: Diagram for the trefoil knot.

We could take the first crossing and smooth it two get two diagrams with one less crossing:


Figure 2: Possible smoothings for the first crossing in trefoil knot.

From the relations of the Jones polynomial, we can see that we only need to understand these two to understand the original one. However, we can apply the same process to each of these and eventually we get the following 8 *resolutions*:

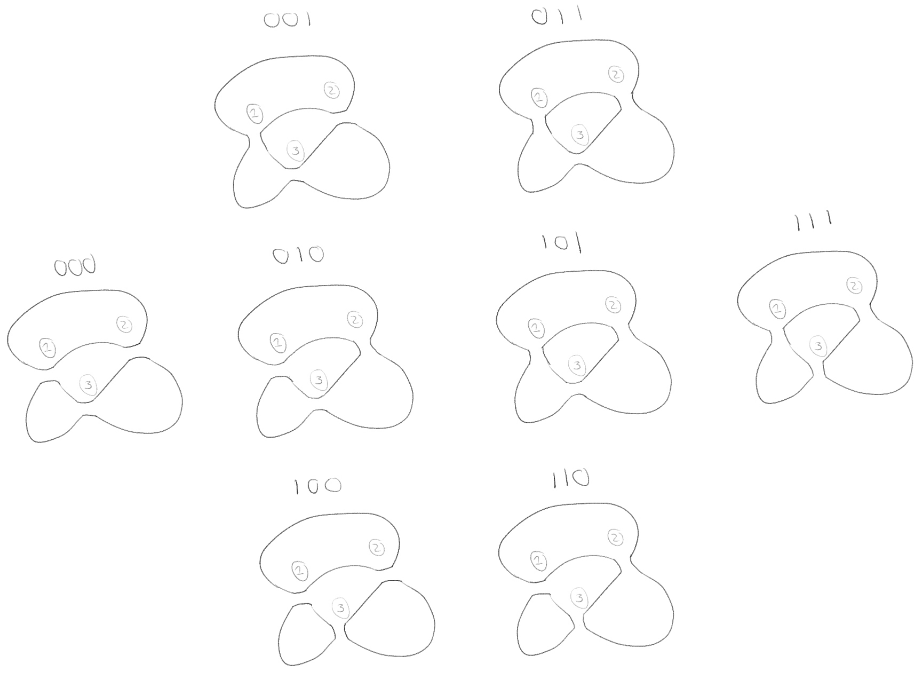


Figure 3: Cube of resolutions for the trefoil knot.

This figure above 3 is what we call the *cube of resolutions*[1] for a knot diagram. It contains all the information we need to compute the Jones polynomial and we can compute it as follows:


Figure 4: Computation for Jones polynomial of trefoil.

Each vertex has a collection of unknots with polynomial $(q+q^{-1})^k$ where $k$ is the number of unknots. The $(-q)^r$ factors come from the number of 1-smoothings we made to obtain each resolution, and it comes from the smoothing relations for the polynomial. Note that we organized these so that the $c^{th}$ column has $c$ 1-smoothings. One can then finish the computation and find the Jones polynomial to be $J(q) = q^2 + q^6 - q^8$.

The power of the Jones polynomial should be clear; the relations give us a way to decrease the number of crossings and so we can always reduce this enough to compute it.

Our goal will be to upgrade the Jones polynomial into a stronger invariant known as **Khovanov homology.** It is possible to go straight ahead and define Khovanov homology directly and prove its invariance. However, the way it originally came up was in the context of TQFTs, and viewing it this way can help us understand more of its properties.

## 2 TQFT

TQFT is an acronym that stands for Topological Quantum Field Theory. The reason TQFTs came about was because they somehow represented physical theories that physicists were interested in.

[1] I'll admit that what I've drawn doesn't look like a cube, but the reason for this is that $n$ crossings give $2^n$ resolutions and an $n$-dimensional cube has $2^n$ corners.

Before, we give the definition of a TQFT, we need to talk a bit about the notion of a *cobordism* between two manifolds.

**Definition 2.1** (Cobordism)**.** A cobordism between two closed, oriented $n$-dimensional manifolds $M_1$ and $M_2$ is an oriented $n+1$-dimensional manifold $N$ such that

$$\partial N = M_1 \sqcup M_2.$$

◇

*Remark* 2.2. It turns out that cobordism gives an equivalence relation between manifolds! Even more, since $M \times I$ gives a cobordism from a manifold to itself. Then if we consider the empty set as a manifold, then it is the identity and the *set* of smooth manifolds up to diffeomorphisms becomes a group under disjoint union; it is even a $\mathbb{Z}/2\mathbb{Z}-$module. If we also consider the product operation, then we get graded ring. ○

The study of cobordisms is very rich and interesting, but we won't get into it here. The definition alone is enough to define TQFTs.

**Definition 2.3** (TQFT)**.** A TQFT of dimension $d$ is an *assignment* $Z$ of a vector space to each diffeomorphism class of closed $d-1$ dimensional manifold, and for each diffeomorphism class of *oriented* cobordism a linear map between the vector spaces. Gluing cobordisms has the effect of composition on the vector spaces. A disjoint union of manifolds should assign the tensor product of the spaces in each of the two manifolds. ◇

*Remark* 2.4. This definition is admittedly a bit funky. It is not at all clear what is meant by an assignment here. This turns out to be pretty technical so we leave a more detailed discussion to the appendix. ○

*Remark* 2.5. It's important that we take diffeomorphism so that we can glue nicely. However, note that this means we should also worry about the choice of smooth structure if we venture into higher dimensions. For low dimensions this doesn't matter. ○

In the vertices of our cube of resolutions we had unknots and the way we related these to build up the diagram was by smoothing the crossings. It turns out that we can think of the unknots as the disjoint union of circles and our smoothings as cobordisms. So what we want to study are 2-TQFTs!

All closed 1 manifolds are disjoint unions of the circle $S^1$ so we can expect that this case could be easy to study. Given a 2-TQFT $Z$, then if we know its value on the circle

$$Z(S^1) = V,$$

then we know its value everywhere

$$Z\Big(\bigsqcup_{i=1}^{k} S^1\Big) = V^{\otimes k}.$$

Now consider the following cobordisms:

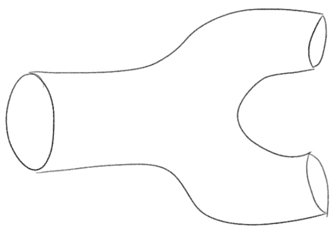

Figure 5: Pair of pants.

**Example** (Pair of Pants)**.** This gives a cobordism betweem $S^1 \sqcup S^1 \to S^1$, and $Z$ takes this to a map

$$\mu : V \otimes V \to V$$

which we call the *multiplication*. We could also think of the pants in the opposite direction, where now instead we get a map

$$\delta : V \to V \otimes V$$

which we call the *comultiplication*.

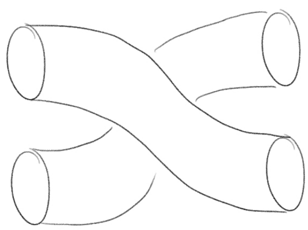

Figure 6: Twist.

**Example** (Twists)**.** We could also consider the twist as above. Notice that if we compose this with the (co)multiplication, we actually get a manifold diffeomorphic to the original so the (co)multiplication must be commutative!

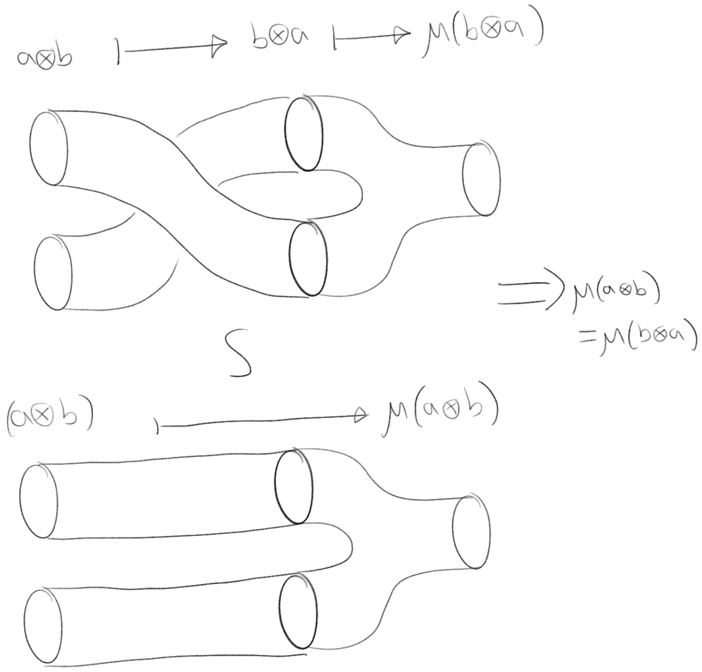


Figure 7: This diffeomorphism shows (co)multiplication is commutative.

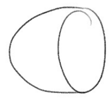

Figure 8: Cap.

**Example** (Disk)**.** Since the circle bounds, this gives a cobordism $S^1 \to \emptyset$, and $Z$ takes this to a map

$$\epsilon : V \to \mathbb{C}$$

which is a unit for the comultiplication. If we look at this cobordism in the opposite direction, then we get a map

$$\eta : V \to \mathbb{C}$$

which gives a unit element for its multiplication. To see that these are indeed units, we can just look at the cobordism below:

**Example** (V is its dual)**.** If we compose these two maps to get $\sigma = \eta \circ \mu$, then we get

$$V \otimes V \to V \to \mathbb{C}$$

which is a non-degenerate pairing and so $V \cong V^*$. This was not immediately obvious at first, but now we can see that $V$ is finite dimensional!

*Exercise* (Frobenius Law). The Frobenius Law states that

$$(1 \otimes \mu) \circ (\delta \otimes 1) = \delta \circ \mu = (\mu \otimes 1) \circ (1 \otimes \delta)$$

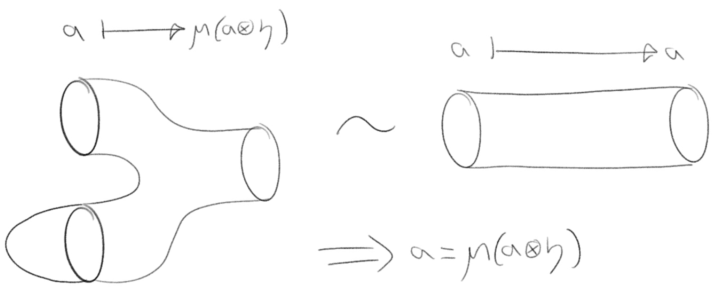


Figure 9: This shows we have a (co)unit.

If you want to gain familiarity with TQFTs, you can try your hand at drawing the cobordisms for these.

**Definition 2.6** (Commutative Frobenius Algebra)**.** A vector space $V$ over a field $k$ with unital multiplication and comultiplication satisfying the Frobenius law is a **Frobenius algebra** (CFA). It is commutative if both the multiplication and comultiplication are commutative. ⋄

**Theorem 2.7.** *A commutative Frobenius algebra is the same as a 2 dimensional TQFT.*

This is great since we now have a purely algebraic characterization of 2 dimensional TQFTs! We've seen how a 2-TQFT provides us with a commutative Frobenius algebra; now we just need to see the other way around.

**Lemma 2.8.** *Any cobordism can be built out of caps, cups and pants.*

I won't make the statement of this more precise, but I'll elaborate a bit on how you can believe in this:

Consider our 2 dimensional manifold $X$ sitting in $\mathbb{R}^3$ and height as a smooth function $h : X \to \mathbb{R}$. Then for any real number $a$, we can consider $X_{\leq a}$ the subset of $X$ which height less than $a$. The topology of $X_{\leq a}$ only changes over critical points of $X$ which are either local extrema or saddles.

*Exercise.* Each of these 3 corresponds to a cap, cup or pant. Which is which?

So if we start at a small value of $a_0$ so that $X_{\leq a_0} = \emptyset$ and go up to some $a_1$ so that $X_{\leq a_1}$, then the way we start building $X$ with the singularities described above.

*Remark* 2.9. There are a lot of details that go into this, and this general approach is called Morse Theory. You can see [MSW69] for more details. ◦

So now we know that what we really have to study are the relations between the pants, caps and cups. It's not at all clear, but you can actually show that the Frobenius algebra gives us all the possible relations between these!

So modulo these details, we've proved 2.7. So if we want to give a 2 TQFT, it's enough to give the Frobenius algebra! The

**Definition 2.10** (Khovanov's TQFT)**.** Let $V$ be the graded $\mathbb{C}$ vector space spanned by the basis $\{v_+, v_-\}$, where these have gradings $\pm q$ corresponding to their subscript. We define the multiplication as

- $\mu(v_+ \otimes v_+) = v_+$
- $\mu(v_+ \otimes v_-) = \mu(v_- \otimes v_+) = v_-$
- $\mu(v_- \otimes v_-) = 0$

and comultiplication as

- $\delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+$
- $\delta(v_-) = v_- \otimes v_-$

◇

The idea will be as follows: in our cube of resolutions 3, all of the vertices were circles and a TQFT assigns to each vertex a vector space. There is even more structure, in fact. We can relate the vertices to one another via cobordisms, and these should induce maps in their vector spaces. The resulting structure will be a chain complex that is homotopy invariant; i.e. its homology is invariant.

# 3 Khovanov Homology

Before going further, we need to talk a bit more seriously about the cube of resolutions and set up some terminology:

**Definition 3.1** (Cube of resolutions)**.** Given a knot diagram $K$, the cube of resolutions is the following data arranged as in figure 3:

- A set of crossings $\chi$ of size $n := |\chi|$ which we number arbitrarily. We also keep track of $n_+$ and $n_-$ the number of positive and negative crossings respectively.
- A set of resolutions corresponding to strings $s \in \{0,1\}^n$ where $s_i$ means the ith crossing is resolved via an $i-$smoothing.
- A set of maps $\xi \in \{0, 1, \star\}^n$ with just one $\star$ given as follows: if $\xi_k = \star$, then for a string $s$ so that $s_k = 0$, we define an map $d_\xi : s \longrightarrow s'$ where $s'_i = s_i$ for all $i$ except at $i = k$ where $s'_k = 1$. That is, the index with $\star$ indicates a shift from 0 to 1.

We arrange the resolutions in $n$ columns as follows: a resolution $s$ belongs in the rth column $C^r$ if $\sum_i s_i = r$, that is if it has exactly r 1-smoothings. Then we can see that the maps in $\xi$ all take an element living in the rth column into the r+1th column. ◇

The cube might seem very scary, but it's actually not all too bad. An odd thing you might have noticed is that, right now, the maps in the cube don't have any meaning. The idea is that we can think of these maps as cobordisms between the knots of the corresponding smoothings.

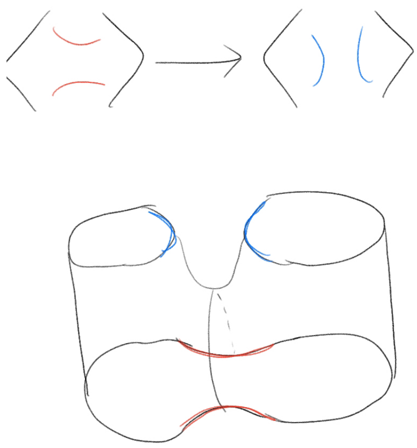

Figure 10: Cobordism between 0 and 1 smoothing.

So then we have collection of circles and cobordisms between these! So now we have arrows in our diagram:

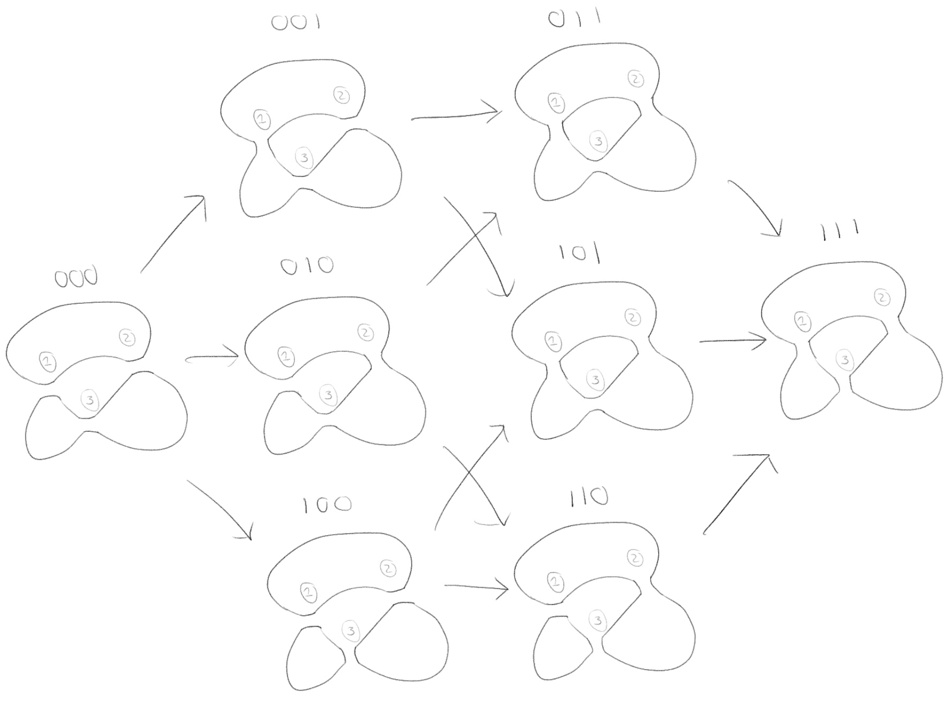


Figure 11: Differentials in cube of resolutions.

The idea still is to apply Khovanov's TQFT to this diagram and eventually get a chain complex. However, there are some nice categorial properties that we have right now and it turns out, we can get "chain complexes" right now by considering formal direct sums, and these are already invariant in a sense.

That is to say, before applying the TQFT to this and showing the homology of the resulting chain complex is invariant, what we have is already a knot invariant in some other sense, and we can just apply Khovanov's TQFT to it. It is worth noting that applying the TQFT to this does lose some information, however, it is a much easier structure to deal with.

We simply skip ahead and define the chain complex of vector spaces that we get:

Let $[\![K]\!]^r$ be the rth column, i.e. resolutions with exactly $r$ 1-smoothings. Then we get

$$[\![K]\!]^n \to ... \to [\![K]\!]^1 \to 0$$

where

$$d_r : [\![K]\!]^r \to [\![K]\!]^{r+1} \text{ is given by } d_r := \sum_{|\xi|=1} (-1)^\xi d_\xi$$

where $(-1)^\xi := \sum_{i<j} \xi_i$ where $j$ is the index at which $\xi_j = \star$ and $|\xi| = \sum_{i \neq j} \xi_i$ which tells us what column it starts on. The map $d_\xi$ itself depends on how the cobordism acts on the vertices and corresponds to either multiplication or comultiplication in Khovanov's TQFT (and it acts by 0 on the other vertices which don't directly participate in this.)

*Exercise.* Show that this satisfies $d_{r+1} \circ d_r = 0$.

This chain complex is still not a knot invariant, but it's homology is, so we can get homology groups $H_i(K)$ where each of these is a graded vector space. We can define the *graded dimension* of a graded vector space $V$ as

$$\text{qdim}(V) := \sum_{n\in\mathbb{Z}} \dim(V_n) q^n$$

where $V_n \subset V$ is the subspace of graded dimension $n$.

Then we can define an invariant based on these:

**Definition 3.2** (Poincaré series)**.** The Poincaré series of graded homology groups $H_i(K)$ is defined as

$$p(t) = \sum_{n\geq 0} t^n \text{qdim}(H_i(K))$$

◇

and this invariant recovers the Jones polynomial:

**Theorem 3.3.** *The Euler Characterstic, defined by $p(-1)$ is the same as the unnormalized Jones polynomial.*

So this invariant indeed recovers the Jones polynomial! This shows that this invariant is at least as strong as the Jones polynomial, and in fact, there are examples that it is indeed stronger (see [BN02].)

# Appendices

## A Chain Complexes

If you haven't seen chain complexes before, here is a quick guide to some basic facts and definitions.

**Definition A.1** (Chain Complex)**.** A **chain complex** is a sequence of vector spaces[2]

$$... \to V_{n+1} \to V_n \to V_{n-1} \to ...$$

with maps $d_n : V_{n+1} \to V_n$ such that $d_n \circ d_{n+1} = 0$. We usually drop the subscript and just write $d^2 = 0$. It is *bounded* if only finitely many groups are non-zero. We will only consider bounded chain complexes. ◇

**Definition A.2** (Homology)**.** The **homology** of a chain complex is defined as

$$H_n := \frac{\ker(d_{n+1})}{\operatorname{im}(d_n)}$$

which is well-defined since the condition $d^2 = 0$ tells us that $\operatorname{im}(d_{n+1}) \subset \ker(d_n)$. ◇

**Definition A.3** (Map of chain complexes)**.** A map of chain complexes is a commutatvive diagram:

$$\begin{array}{ccccccccc} ... & \longrightarrow & V_{n+1}, & \longrightarrow & V_n, & \longrightarrow & V_{n-1}, & \longrightarrow & ... \\ & & \downarrow & & \downarrow & & \downarrow & & \\ ... & \longrightarrow & W_{n+1} & \longrightarrow & W_n & \longrightarrow & W_{n-1} & \longrightarrow & ... \end{array}$$

◇

*Remark* A.4. Such a diagram induces a well-defined map in homology. If the map is an isomorphism in homology, we say this is a *quasi-isomorphism.* ○

[2]More generally, you could consider $R$-modules for a commutative ring $R$. A common example would be abelian groups taking $R = \mathbb{Z}$.

# B A bit about categories

Some of the definitions which we've seen here are secretly related to category theory. We briefly explain some of the relations.

**Definition B.1** (Category)**.** A category $\mathcal{C}$ is the following data:

- A collection of objects $\mathrm{Ob}(\mathcal{C})$
- A set of morphisms $\mathcal{C}(x, y)$ for any two objects $a$ and $b$.
- A composition law

◇

Here are the main examples for us

**Definition B.2** (Cobordism Category)**.** We can consider d-dimensional (oriented) cobordisms to be a category in the following way:

- It has closed $d-1$ dimensional manifolds (up to diffeomorphism) as objects
- It has oriented $d$ dimensional manifolds (up to diffeomorphism) as morphisms between its boundary components.

◇

**Definition B.3** (Vector Spaces)**.** We can consider vector spaces over a given field, say $\mathbb{C}$, as a category where

- It has vector spaces as objects
- It has $\mathbb{C}-$linear maps as morphisms.

◇

So we can see that TQFT is related to these categorical notions: it is some sort of function between categories. This turns out to be what is called a functor.

**Definition B.4** (Functor)**.** A functor $F : \mathcal{C} \to \mathcal{D}$

- Associates to each object $X \in \mathrm{Ob}(\mathcal{C})$ an object $F(X) \in \mathrm{Ob}(\mathcal{D})$
- Associates to each $f \in \mathcal{C}(X, Y)$ a morphism $F(f) \in \mathcal{D}(F(X), F(Y))$
- The assignment of morphisms must take the identity morphism to the identity morphism and it must respect composition of morphisms

◇

So now we can (almost) formalize our definition of TQFT:

**Definition B.5** (TQFT)**.** A TQFT is a symmetric monoidal functor $Z : \mathbf{Cob}(n) \to \mathbf{Vect}_{\mathbb{C}}$. ◇

What does symmetric monoidal mean? A symmetric monoidal category is a category equipped with a tensor product of sorts, and a symmetric monoidal functor is one that respects this. The symmetric monoidal structure for vector spaces is the tensor product and for cobordisms is the disjoint union.

*Remark* B.6. There exists a notion of what is called a pre-additive category for which it makes sense to define a notion of complexes. The nature of the cube of resolutions is related to this, and one could get a "chain complex" for the cobordism category. Then, instead of applying Khovanov's TQFT to this and showing that it's a knot invariant, one could show that this is already an invariant in some sense and that Khovanov's TQFT preserves it. In particular, this is the step where Khovanov's TQFT itself matters, and any other TQFT that preserves this notion of invariance would also work. ○

## References

[BN02] Dror Bar-Natan. On khovanov's categorification of the jones polynomial. *Algebraic &amp Geometric Topology*, 2(1):337–370, may 2002. 10

[MSW69] J. Milnor, M. SPIVAK, and R. WELLS. *Morse Theory. (AM-51), Volume 51*. Princeton University Press, 1969. 7