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Lecture Three (Hechen Hu, Stack of Formal Group Laws, 9/16/25)

Definition. A *formal group law* over a ring R is a formal power series $f(x, y) \in R[\![x, y]\!]$ such that

1. $f(x, 0) = f(0, x) = x$;
2. $f(x, y) = f(y, x)$;
3. $f(x, f(y, z)) = f(f(x, y), z)$;

Set FGL to be the functor assigning R the set of formal group laws over R.

If $f(x, y) = \sum_{i,j \geqslant 0} c_{ij} x^i y^j$, then the three conditions imposes polynomial relations on c_{ij}, e.g. the first condition says that $c_{i0} = c_{0i} = \delta_{i1}$. The *Lazard ring* L is the quotient $\mathbb{Z}[c_{ij}]/Q$ by these relations. It corepresents the functor FGL by the previous discussion. If we grade $\mathbb{Z}[c_{ij}]$ by $\deg(c_{ij}) = 2(i + j - 1)$ and set the variables x, y, z to have degree -2, then $f(x, y) = \sum_{i,j} c_{ij} x^i y^j$, $f(x, f(y, z))$, and $f(f(x, y), z)$ all have degree -2. As a consequence, the coefficients of $x^i y^j z^k$ in both $f(x, f(y, z))$ and $f(f(x, y), z)$ have degree $2(i + j + k) - 2$, i.e. Q is generated by homogeneous elements. Thus the grading descends to a nonnegative grading on L with $L_0 = \mathbb{Z}$.

Theorem 2.0.1 (Lazard)**.** *L is a polynomial ring $\mathbb{Z}[t_1, t_2, \cdots]$ in infinitely many variables with $\deg(t_i) = 2i$.*

2.1 Stack of Formal Group Laws

Let G^+ be the group scheme defined by $G^+(R) = \{b_0 x + b_1 x^2 + \cdots : b_0 \in R^*\} \subset R[\![x]\!]$. One can think of this as the group of coordinate substitutions $x \mapsto b_0 x + b_1 x^2 + \cdots$. It has an action on $\operatorname{Spec} L$ that is induced by applying substitution to formal group laws, i.e. sending a formal group law $f \in \mathrm{FGL}(R)$ to the formal group law $g(f(g^{-1}(x), g^{-1}(y)))$. Let $G \subset G^+$ be the subgroup consisting of elements with $b_0 = 1$.

Definition. The *moduli stack of formal groups* is $\mathcal{M}_{FG} := [\operatorname{Spec} L/G^+]$. The *moduli stack of formal groups and strict isomorphisms* is $\mathcal{M}^s_{FG} := [\operatorname{Spec} L/G]$.

By definition, objects in $\mathcal{M}_{FG}(\operatorname{Spec} R)$ are principal G^+-bundles $P \to \operatorname{Spec} R$ with an G^+-equivariant morphism $P \to \operatorname{Spec} L$. Examining a trivialization of the bundle P over a covering $\{\operatorname{Spec} R_i \to \operatorname{Spec} R\}$ shows that the object correspond to a collection of formal group laws over R_i up to the action by G^+, i.e. up to substitution. Thus the objects are not formal group laws over R as we defined (which do not satisfy descent) but what Lurie defined in Definition 5 of Lecture 11. Our formal group laws over R are called *coordinatizable* by Lurie. The morphisms in $\mathcal{M}_{FG}(\operatorname{Spec} R)$ are isomorphisms of principal G^+-bundles by definition.

Similarly, $\mathcal{M}^s_{FG}$ parametrizes formal group laws up to substitution by elements of G. These substitutions are called *strict isomorphisms* because they do not change the first order part of the coordinate. In fact we have better: the group G is an iterated extension of copies of the additive group, which has no cohomology over affine schemes (they are quasi-coherent), hence all principal G-bundles over $\operatorname{Spec} R$, which is classified by $H^1(\operatorname{Spec} R, G)$, are trivial. Thus $\mathcal{M}^s_{FG}$ parametrizes coordinatizable formal groups.

Remark. The canonical map $\mathcal{M}^s_{FG} \to \mathcal{M}_{FG}$ is a principal $\mathbb{G}_m$-bundle (the fibers are G^+/G, which is isomorphic to $\mathbb{G}_m$ because any two substitution with the same b_0 are in the same G-orbit). Thus there is an associated line bundle ω on $\mathcal{M}_{FG}$ with $\mathcal{M}^s_{FG}$ parametrizing trivializations of ω. In fact ω is the line bundle of invariant differentials on the formal groups.

2.2 Connection to Homotopy Theory

Quillen's theorem says that the map $L \to \pi_* MU$ corresponding to the universal complex orientation is an isomorphism between graded rings.

Proposition (Relevant Results)**.** Let E be a complex oriented cohomology theory, MU the complex cobordism spectrum, and X any spectrum.

1. There is an isomorphism $\pi_*(MU \otimes E) = E_*(MU) \cong (\pi_* E)[b_1, b_2, \cdots]$, which is the ring of functions on $G \times \pi_*(E)$;
2. The composition $L \to \pi_*(MU) \to H_*(MU; \mathbb{Z}) \cong \mathbb{Z}[b_1, b_2, \cdots]$ with the Hurewicz map correspond to the formal group law defined by $(x, y) \mapsto g(g^{-1}(x) + g^{-1}y)$, where $g(x) = x + b_1 x^2 + b_2 x^3 + \cdots$;
3. The $\pi_*(X)$-module $MU_*(X)$ admits a G-action that is compatible with the grading. Similarly, $MU_{\text{even}}(X)$ and $MU_{\text{odd}}(X)$ admit a G^+-action that is compatible with the gradings.

The last assertion says that $MU_*(X), MU_{\text{even}}(X), MU_{\text{odd}}(X)$ gives quasi-coherent sheaves $\mathcal{F}_X, \mathcal{F}_X^{\text{even}}, \mathcal{F}_X^{\text{odd}}$ on the corresponding quotient stacks.

Theorem 2.2.1 (Adams-Novikov Spectral Sequence)**.** *For any spectrum X, there is a spectral sequence $\{E_r^{p,q}, d_r\}$, converging to a finite filtration of $\pi_{p-q}(X)$ if X is connective. The E_2-page is given by*

$$E_2^{2a,b} = H^b(\mathcal{M}_{FG}; \mathcal{F}_X^{even} \otimes \omega^a), \quad E_2^{2a+1,b} = H^b(\mathcal{M}_{FG}; \mathcal{F}_X^{odd} \otimes \omega^a)$$