Limits & (Co)limits

Preliminaries:
Recall joins of simplicial sets allow us to define slice ∞-categories as adjoints to functors

$$S \star - : sSet \longrightarrow sSet_{S/}$$

From this we are able to define initial & final objects
↳ $\mathcal{C}$ an ∞-category, $x \in \mathcal{C}$ is a final object if the forgetful map
$\mathcal{C}_{/x} \longrightarrow \mathcal{C}$ is an acyclic fibration of sSets.

Recall (ordinary category theory):
if $p : A \to B$ is a functor (typically A is an indexing category), the colimit of p is an object $\operatorname{colim}_A p$ in B, with a universal cocone (i.e. a natural transformation $\pi : p \longrightarrow \underline{1}_{\operatorname{colim}_A p}$

constant diagram from A to $\operatorname{colim}_A p$

which is universal (that is, for any other such object $b \in B$ with a natural transformation $\pi' : p \to \underline{1}_b$, there is a morphism $\operatorname{colim}_A p \to b$ in B making the obvious diagram commute)).

Equivalently, letting $B_{\rho/}$ denote the slice category under ρ (also the cocone category of ρ), $\operatorname{colim}_A \rho$ will be an initial object in $B_{\rho/}$.

Since we have the technology of slice categories & initial objects for ∞-categories, we can readily define,

Def^n Let K be any simplicial set, & $\mathcal{C}$ an ∞-category. For any morphism $\rho: K \to \mathcal{C}$ of simplicial sets (a.k.a diagram) the colimit $\operatorname{colim}_{\mathcal{C}} \rho$ of ρ will be an initial object in the slice ∞-category $\mathcal{C}_{\rho/}$.

$\mathcal{C}$ is called cocomplete if it contains all colims of small diagrams.

Note that if there is an initial object in $\mathcal{C}_{\rho/}$, then the full ∞-subcategory of initial objects in $\mathcal{C}_{\rho/}$ is a contractible Kan complex

↳ i.e. any $\sigma: \partial\Delta^n \to (\mathcal{C}_{\rho/})_{in}$ can be extended to $\bar{\sigma}: \Delta^n \to (\mathcal{C}_{\rho/})_{in}$

(which, for $n > 0$, is an essentially equivalent definition for $c \in \mathcal{C}_{\rho/}$ being an initial object).

So, if a colimit exists, it is unique up to contractible choice.

Another (equivalent) way:

Def^n (Krause-Nikolaus)

$F: K \to \mathcal{C}$ functor (I think, really, ∞-cats) & define

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(F, x) \arrow[r] \arrow[d, "\mathcal{J}'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(K * \Delta^0, \mathcal{C}) \arrow[d, "\mathcal{J}"] & \pi \arrow[d, mapsto] \\ \Delta^0 \arrow[r, "(F, x)"] & \mathrm{Fun}(K, \mathcal{C}) \times \mathcal{C} & (\mathrm{res}_K \pi, \pi(\infty)) \end{tikzcd}$$

Note, $\mathcal{J}$ is a restriction along an inclusion, & hence is a conservative inner fibration (which is preserved under pullback).

Rmk if $F = y: \Delta^0 \to \mathcal{C}$, then
$\mathrm{Map}_{\mathcal{C}}(F, x) = \mathrm{map}_{\mathcal{C}}(y, x)$

Useful facts
$F: K \to \mathcal{C}$ & $i: L \to K$ maps of simplicial sets.
Then, there is a map
$$\mathrm{Map}_{\mathcal{C}}(F, x) \to \mathrm{Map}_{\mathcal{C}}(F \circ i, x)$$
which is a homotopy equivalence if i is right-anodyne.

~~Rmk~~
Def^n Let $F: K \to \mathcal{C}$ be a functor, &
$\bar{F}: K * \Delta^0 \to \mathcal{C}$ a cone/F (i.e. the natural restriction of $K \hookrightarrow K * \Delta^0$ recovers F).
$\bar{F}$ is a colimit cone if for all $x \in \mathcal{C}$,
$\mathrm{Map}_{\mathcal{C}}(\bar{F}, x) \to \mathrm{Map}_{\mathcal{C}}(F, x)$ is a htpy equivalence.

Note that since $\{\infty\} \hookrightarrow K * \Delta^0$ is right-anodyne, we have,

$$\mathrm{Map}_{\mathcal{C}}(F, x) \underset{\text{htpy}}{\simeq} \mathrm{Map}_{\mathcal{C}}(\bar{F}(\infty), x) = \mathrm{map}_{\mathcal{C}}(\bar{F}(\infty), x)$$

So, $\bar{F}$ being a colimit cone is the same as

$$\mathrm{Map}_{\mathcal{C}}(F, x) \underset{\text{htpy}}{\simeq} \mathrm{map}_{\mathcal{C}}(\bar{F}(\infty), x)$$

[this is what we expect from colimits].

Examples

(i) (co)limit over a set (~~as~~ as discrete category) is (co)product

(ii) colimit of $\Lambda^2_0 \to \mathcal{C}$ is a pushout, & limit of $\Lambda^2_2 \to \mathcal{C}$ is a pullback.

Both constructions of limits match-up.

We note here, ~~as it will~~ for the sake of edification, that

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(F, x) \arrow[r] \arrow[d] & \mathcal{C}_{F/} \arrow[d] \\ \Delta^0 \arrow[r, "x"] & \mathcal{C} \end{tikzcd}$$

left-fibration

straightening-unstraightening

$f : \mathcal{C} \to \mathrm{Spc}$

$x \longmapsto \mathrm{Map}_{\mathcal{C}}(F, x)$

is homotopy-cartesian.

We say, $f: \mathcal{C} \to \mathcal{S}pc$ is representable if it is equivalent to the functor ~~map~~ $map_{\mathcal{C}}(x, -)$ for some x in $\mathcal{C}$.

$\Leftrightarrow$ associated left-fibration $\begin{matrix} \mathcal{E} \\ \downarrow \\ \mathcal{C} \end{matrix}$ is equivalent to $\begin{matrix} \mathcal{C}_{x/} \\ \downarrow \\ \mathcal{C} \end{matrix}$.

We can use representability to characterize the existence of colimits as follows,

Prop^n

Let $F: K \to \mathcal{C}$ be a functor. F admits colimits if & only if $Map_{\mathcal{C}}(F, -)$ is representable, & any rep. object is a colimit of F.

Pf Sketch

($\Rightarrow$) Let $\bar{F}: K * \Delta^0 \to \mathcal{C}$ be a colimit cone, w/ $x = \bar{F}(\infty)$.

Then,

$$\mathcal{C}_{F/} \xleftarrow{a} \mathcal{C}_{\bar{F}/} \xrightarrow{b} \mathcal{C}_{x/}$$

are both equivalences of left fibrations (a, by def^n of $\bar{F}$ being a colimit cone, & b b/c $\{\infty\} \hookrightarrow$ ~~$K * \{\infty\}$~~ $K * \Delta^0$ is right anodyne).

Thus, ~~F is re~~ $Map_{\mathcal{C}}(F, -)$ is representable by x.

($\Leftarrow$) Assume $Map_{\mathcal{C}}(F, -)$ is representable by an object x.

Thus, there is an equivalence of left fibrations

$$\begin{tikzcd} \mathcal{C}_{x/} \arrow[r, "\simeq"] \arrow[d] & \mathcal{C}_{F/} \arrow[d] \\ \mathcal{C} & \mathcal{C} \end{tikzcd}$$

Take $x \xrightarrow{\mathrm{id}} x$ in $\mathcal{C}_{x/}$. This is an initial object in $\mathcal{C}_{x/}$, & hence is mapped to an initial object in $\mathcal{C}_{F/}$, which is $\operatorname{colim}_{\mathcal{C}} F$, & which is equal to x.

$\square$.

This gives us a whole host of results.

<u>Ex</u> $F : \Delta^n \to \mathcal{C}$ admits both colimits & limits.

↳ $$\begin{tikzcd} \mathcal{C}_{F/} \arrow[r, "\simeq"] \arrow[d] & \mathcal{C}_{F(n)/} \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

<u>Some big hits</u>

(1) If $\mathcal{C}$ an ∞-category, admits small coproducts & small pushouts, then it admits all small colimits.

↳ see Lurie for full details, but really an induction on $\dim K$. argument over finite indexing simplicial sets via skeletal pushout

$$\begin{tikzcd} \coprod_{i \in I} \partial\Delta^n \arrow[r] \arrow[d] & sk_{n-1}(K) \arrow[d] \\ \coprod_{i \in I} \Delta^n \arrow[r] & K \end{tikzcd}$$

& a "3 out of 4" pushout-colimit argument.

Remark / Def^n A functor $f : \mathcal{C} \to \mathcal{D}$ preserves K-shaped colimits if for every functor $F : K \to N^{\star}\mathcal{C}$, the induced func.

$$\mathcal{C}^{F/} \longrightarrow \mathcal{D}^{f \circ F/}$$

sends initial objects to initial objects.

Fact The ∞-categories Cat_∞ & Spc admit all small limits & colimits.

↳ two proofs

↳ one, using straightening-unstraightening

Note all of these statements have analogues for ordinary categories (indeed, the Nerve functor is fully faithful, & $N(\mathcal{C}_{F/}) = N(\mathcal{C})_{N(F)/}$).

Some words about cofinal/coinitial functors.

Def^n Let $f : K \to L$ map of simp. sets, &

$p : X \to L$ an inner fibration.

Define $\mathrm{Fun}_f(K, X)$ to be the pullback

$$\begin{tikzcd}
\mathrm{Fun}_f(K,X) \arrow[r] \arrow[d] & \mathrm{Fun}(K,X) \arrow[d, "p_*"] \\
\Delta^0 \arrow[r, "f"] & \mathrm{Fun}(K,L)
\end{tikzcd}$$

(informally, $\mathrm{Fun}_f(K,X)$ ~~cons~~ consists of functors $K \to X$ which, when you postcompose with p, you f.

Def^n $f: K \to L$ is <u>cofinal</u> if for all $p: X \to L$ right fibr., any morphism $\pi: K \to X$ which becomes f really is the same as a morphism $\pi': L \to X$ which becomes the identity.

Formally,

$$\mathrm{Fun}_{id}(L, X) \xrightarrow{f^*} \mathrm{Fun}_f(K, X)$$

is a Joyal equivalence.

Similarly, f <u>coinitial</u> if for all $p: X \to L$ left fibr. ...

<u>Put (*) here</u>

Def^n Let $p: Y \to X$ be a map of simp. sets.

We say p is <u>smooth</u> if for all pullback diagram

$$\begin{array}{ccc} B & \xrightarrow{j} & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow[i]{} & X \end{array}$$

i cofinal $\Rightarrow$ j cofinal.

p is <u>proper</u> if i coinitial $\Rightarrow$ j coinitial.

(has to do with base change theorems in étale coh., see Lurie).

Prop^n (Facts) LFibs (in fact, any ~~cofib~~ CoCar) are universally smooth, & RFibs (in fact, any Car) are universally proper.

Thm

$f: \mathcal{C} \to \mathcal{D}$ map of simplicial sets w/ $\mathcal{D}$ an ∞-category.
Then, f cofinal iff. for all $d \in \mathcal{D}$,
forming the pullback

$$\begin{tikzcd} \mathcal{C}_{d/} \arrow[r, "\tilde{f}"] \arrow[d] & \mathcal{D}_{d/} \arrow[d, "p"] \\ \mathcal{C} \arrow[r, "f"] & \mathcal{D} \end{tikzcd}$$

$\mathcal{C}_{d/}$ is weakly contractible.

Pf sketch of only if

Assume f cofinal.
p is a left fibration, hence smooth.
Thus, $\tilde{f}$ is cofinal, & hence a weak equivalence of ~~∞-categories~~ simplicial sets.
Since $\mathcal{D}_{d/}$ has an initial object, it is weakly contractible (its geometric realization contracts to the point of the initial object).
So, $\mathcal{C}_{d/}$ is weakly contractible.

$\square$.

(*) If $v: K' \to K$ is a map of (small) simplicial sets then v is cofinal iff. for any ∞-category $\mathcal{C}$ w/ $p: K \to \mathcal{C}$ a diagram, $p' = p \circ v : K' \to \mathcal{C}$ the induced diagram, the natural map of ∞-cats

$$\mathcal{C}_{p/} \to \mathcal{C}_{p'/} \text{ is an equivalence.}$$

Corollary (Quillen's Theorem A).

$f: \mathcal{C} \to \mathcal{D}$ functor b/w ordinary categories.
For all $D \in \mathcal{D}$, the fiber product category
$$\mathcal{C}_{D/} := \mathcal{C} \times_f \mathcal{D}_{D/}$$ has weakly contractible nerve implies

f induces weak htpy equiv. of simp. sets
$N(f): N(\mathcal{C}) \to N(\mathcal{D})$.

Pf: Trivial.

What can we do with colimits?

If $u: \mathcal{C}_0 \subset \mathcal{C}$ is a full ~~small~~ small subcategory, & $\mathcal{D}$ is another ~~infinity~~ category, we have
$$u^*: \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_0, \mathcal{D}).$$

LK_u is a left-adjoint to this.

haven't introduced this yet

$$LK_u: \mathrm{Fun}(\mathcal{C}_0, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$
left Kan extension.

Recall, in ordinary categories, if $\mathcal{D}$ is cocomplete, we can compute LK_u pointwise as, for $F: \mathcal{C}_0 \to \mathcal{D}$,
$$\underset{\in \mathcal{C}}{(LK_u(F))(x)} = \operatorname{colim}_{i \in \mathcal{C}_{0/x}} F(c).$$

This is mirrored in ∞-cats as follows,

$$\begin{tikzcd}
\mathcal{C}^0 \arrow[r, "F_0"] \arrow[d, hook, "\text{inclusion as full subcat}"'] & \mathcal{D} \arrow[d, "p", "\text{inner fibration}"'] \\
\mathcal{C} \arrow[r] \arrow[ur, "F"] & \mathcal{D}'
\end{tikzcd}$$

F is a p-left Kan ~~fibration~~ extension of F_0 at $C \in \mathcal{C}$ if

$$\begin{tikzcd}
(\mathcal{C}^0_{/C}) \arrow[r, "F_{0/C}"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\
(\mathcal{C}^0_{/C})^{\triangleright} \arrow[r] \arrow[ur] & \mathcal{D}'
\end{tikzcd}$$

gives $F(C)$ as ~~colim~~ p-colimit of $F_{0/C}$.