Etale Seminar #1

Weil Conjectures

Central Problem: counting points on algebraic varieties, specifically K-points for a field K.
In our case, K finite field, say $K=\mathbb{F}_q$, $q=p^n$.

For us, $X/\mathbb{F}_q$ algebraic variety (a.k.a finite type, separated & geom. integral)

Want to understand behaviour of $\mathbb{F}_{q^n}$-points of X for $n\in\mathbb{N}$. Do this all together for all $n\in\mathbb{N}$, via Zeta Function

$$\zeta_X(t) = \exp\Big(\sum_{n\in\mathbb{N}} \frac{X(\mathbb{F}_{q^n})}{n} t^n\Big).$$

Conj. (now theorems)

1) $\zeta_X(t)$ is a rat'l function
2) Let X be smooth & proper of dim. n.

$$\zeta_X(q^{-n}t^{-1}) = \pm q^{\frac{nE}{2}} t^E \zeta_X(t),$$

where E is Euler characteristic.

3) (RH) All roots & poles of $\zeta_X(t)$ have abs. value $q^{\frac{i}{2}}$ for $i\in\mathbb{Z}$.

4) Let X be smooth & proper.
The number of roots & poles w/ abs. value $q^{-i/2}$ is equal to ith Betti number of $X_{\overline{\mathbb{F}}_q}$

How to prove something like this?

Some key characteristics of a Weil cohomology theory: (for us)

L a field, X smooth, proj. variety, of dim n / L.

i) $H^i(X)$ is a fin. dim vector space / L, $i = 0, 1, 2, \dots$
↳ char $L = 0$

2) $H^i(X) = 0$ for $i > 2n$

3) (Lefschetz Fixed-Point Formula) Let $\phi : X \to X$ morphism w/ finitely many fixed points of mult. 1 (i.e. $\Gamma(\phi) \cap \Gamma(id) = A$ transversal finite intersection). Then,

$$\#A = \sum_{i=0}^{2n} (-1)^i \, Tr(\phi \mid H^i(X))$$

4) Poincaré duality.

Let us imagine that we had a Weil cohomology theory for varieties over finite fields, say $H^i(X)$. for $X/\mathbb{F}_q$
↳ smooth, proj.

Letting F be Frobenius, we have that $X(\mathbb{F}_{q^n})$ is the set of fixed points of F^n.

Supposing F only has one eigenvalue/eigenvector α_i on $H^i(X)$ (so $H^i(X)$ is 1-dimensional).

Then, $\#X(\mathbb{F}_{q^m}) = \sum_{i=0}^{2n} (-1)^i \alpha_i^m$, & so simplifying

$$\zeta_X(t) = \prod_{i=0}^{2n} (1-\alpha_i t)^{(-1)^{i+1}}$$, which is rat'l.

The case for multiple eigenvalues of arbitrary multiplicity is similar.
So, we would immediately get that the zeta function is rat'l using fin. dim. of $H^i(X)$ & the Lefschetz fixed-point theorem.

The funct'l eq^n would follow from Poincaré duality
:

Okay so can we just create a Weil cohomology theory?
Sheaf cohomology does not work naively! Flasque sheaves.
Difficulties No Weil cohomology theories over $\mathbb{Q}$. (Serre)
↳ Let E be supersingular elliptic curve. Then
$\mathrm{End}(\bar{E}) \otimes \mathbb{Q}$ is a non-split Quaternion algebra $/\mathbb{Q}$.
functoriality + Kunneth ⇒ $\mathrm{End}(E) \circlearrowright H^1(E)$
↳ need to check for $f, g \in \mathrm{End}(E)$ & $\alpha \in H^1(E)$,
$(f+g)(\alpha) = f(\alpha) + g(\alpha)$.
Note $H^1(E) \xrightarrow{\Delta} H^1(E) \oplus H^1(E)$ is induced

Kunneth

from $E \times E \xrightarrow{+} E$, so taking

$$\begin{tikzcd} E \arrow[r, "{(f,g)}"] \arrow[rr, bend right, "f+g"'] & E \times E \arrow[r, "+"] & E \end{tikzcd}$$

get
$$\begin{aligned} H^1(E) &\longrightarrow H^1(E) \\ \alpha &\longmapsto f(\alpha)+g(\alpha). \end{aligned}$$

So, get $\mathrm{End}(E) \otimes \mathbb{Q} \longrightarrow \mathrm{End}_{\mathbb{Q}}(H^1(E)) = M_2(\mathbb{Q})$.
No such homs exist!

$\square$

Similar ideas show that there is no Weil cohomology theory for $\mathbb{Q}_p$, $p \mid \operatorname{char} \mathbb{F}_q$.

We can make it "work" w/ $\mathbb{Q}_\ell$-coefficients, $\ell \neq p$
↳ i.e. get some of the desired properties of a Weil cohomology theory
using étale.

Étale Maps & Morphisms

Def^n $f: X \longrightarrow Y$ morph of schemes is étale if it is locally of fin. presentation, flat, & unramified.

TFAE for f loc. of fin. presentation & flat.

(i) $\Omega^1_{X/Y} = 0$.

(ii) all residue field extensions are separable, $f^*(m_y)\mathcal{O}_{X,x} = m_x$ ($y = f(x)$) & induced res. field map is separable

(iii) smooth of rel. dim. 0

(iv) formally étale, i.e. for any nilpotent ideal $I \subseteq A$, the lifting problem

$$\begin{tikzcd}
\operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists !"] & Y
\end{tikzcd}$$

has a unique sol^n.

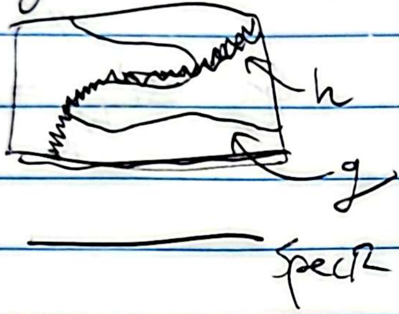
(lifting tangent vectors & infinitessimal thickenings)

(v) locally standard étale:

for each $x \in X$, $y = f(x)$, can find $\mathcal{U} \ni x$ & $V \ni y$ s.t. $f(\mathcal{U}) \subset V$

$V = \operatorname{Spec} R$ $\qquad$ $\mathcal{U} = \operatorname{Spec} R[x]_h/g$

w/ g' a unit in $R[x]_h$ & g monic.



Examples & non-examples.

- $[n] : E \longrightarrow E$ any abelian variety where ~~$n \nmid \operatorname{char} k$~~ $(n, \operatorname{char} k) = 1$

- $\mathbb{G}_m \xrightarrow{f} \mathbb{G}_m$ if ~~$n \nmid \operatorname{ch}$~~ $(n, \operatorname{char} k) = 1$.

  "$t \longmapsto t^n$"

  $$\Omega^1_{X/Y} = k[t, t^{-1}]dt \big/ \underset{\text{invertible}}{nt^{n-1}} = 0.$$

- Any open immersion (so étale maps are not necessarily proper).

- $\mathbb{G}_m \setminus \{1\} \xrightarrow{x \mapsto x^2} \mathbb{G}_m$. ($\operatorname{char} k \neq 2$).

  étale surjection but not proper.
  (so étale maps are not necessarily proper onto their image).

- L/K , $\operatorname{Spec} L \longrightarrow \operatorname{Spec} K$ finite separable extension

$L = K(\alpha)$, α satisfying

$p(x) = 0$ monic

$(p(x), p'(x)) = 1$. w/ $p'(\alpha) \neq 0$.

So $\Omega^1_{L/K} = K(\alpha)\, d\alpha / p'(\alpha) = 0$

In fact, this is an iff.

- (Non-example).

$$\mathbb{A}^1 \xrightarrow{x \mapsto x^2} \mathbb{A}^1$$ not étale (ramified at 0).

$$\Omega^1_{X/Y} = k[t]dt/2t$$

- (Non-example)

Frobenius.

Properties of étale maps

(i) Open immersions are étale.

(ii) Composition of étale maps is étale

(iii) Base change of étale maps is étale.

(iv) (2 out of 3 of 2 out of 3).

If $X \xrightarrow{f} Y \xrightarrow{g} Z$ & g & $g\circ f$ are étale, then f is étale (use formally étale criterion).