ETALE

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An attempt to learn more about etale cohomology. I will use the spelling "etale" due to my laziness.

1. Introduction

Lecture is given by Vidhu.

Let X be a variety over $k = \mathbf{F}_q$ a finite field, where $q = p^m$. We want to understand the $\mathbf{F}_q$-points of X . The idea is to look at $X(\mathbf{F}_{q^n})$ for all n at once. So let

$$\zeta_X(t) = \exp\left(\sum_n \frac{\#X(\mathbf{F}_{p^n})}{n} t^n\right).$$

This is the zeta function for X. Weil had the following conjectures, which are now theorems:

(1) $\zeta_X(t)$ is a rational function;

(2) There is a functional equation: if X is smooth and proper of dimension n, then

$$\zeta_X(q^{-n}t^{-1}) = \pm q^{\frac{nE}{2}} t^E \zeta_X(t)$$

where E is the Euler characteristic;

(3) (Riemann Hypothesis) All roots and poles of $\zeta_X(t)$ have absolute value $q^{\frac{i}{2}}$, $i \in \mathbf{Z}$;

(4) If X is smooth and proper, the number of roots and poles with absolute value $q^{-\frac{i}{2}}$ is equal to the i-th Betti number.

To prove this, Weil tried to developed the Weil cohomology theories. Let L be a field of characteristic 0, X a smooth projective variety over some field k. Weil cohomology theories assigns to X some vectors spaces over L denoted by $H^i(X)$ for each $i \geq 0$, such that

(1) H^i is functorial: if $X \to Y$ is a morphism of varieties, then we have a linear map $f^* : H^i(Y) \to H^i(X)$;

(2) $H^i(X) = 0$ for $i > 2n$;

(3) The Lefschetz fixed point formula should be true: if $\varphi : X \to X$ has finitely many fixed points, each of multiplicity 1, then

$$\#X^\varphi = \sum_{i=0}^{2n} (-1)^i \operatorname{tr}(\varphi^* : H^i(X) \to H^i(X))$$

where X^φ is the set of fixed points;

(4) The Kunneth formula should be true;

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(5) The Poincare duality should be true.

What is this good? Suppose we have a Weil cohomology theory for smooth projective varieties over $k = \mathbf{F}_q$. Let $F : X \to X$ be the Frobenius, then $X(\mathbf{F}_{q^n}) = X^{F^n}$. Assume for now F^* has only one eigenvalue, so under some basis on H^i which is one dimensional it is just multiplication by some α_i. Then the Lefschetz fixed point formula says

$$\#X(\mathbf{F}_{q^n}) = \sum_{i=0}^{2n} (-1)^i \alpha_i^n$$

so

$$\zeta_X(t) = \prod_{i=0}^{2n} (1 - \alpha_i t)^{(-1)^{i+1}}.$$

This says the zeta function is rational. The functional equation will also follow from the Poincare duality.

Question: given a field L of characteristic zero, can we create a Weil cohomology theory? Sheaf cohomology for any sheaf won't work, since non-trivial sheaves give vector spaces over $k = \mathbf{F}_q$, not L. We could try the constant sheaf $\underline{L}$ but then cohomology vanishes. In fact, Serre showed that there is no cohomology theorem for $L = \mathbf{Q}$ such that it is functorial, satisfies the Kunneth formula, and $H^1(C) = \mathbf{Q}^2$ for C an elliptic curve. The idea is to look at a supersingular elliptic curve E. A similar argument shows that there is no Weil cohomology theory over $\mathbf{Q}_p$, where $p = \mathrm{char}(\mathbf{F}_q)$.

However, we can make it work over $\mathbf{Q}_l$ where l doesn't divide p using etale cohomology.

2. Etale rings maps

Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be noetherian local rings. A local ring homomorphism (or as I like to call it, a local map of local rings) $f : A \to B$ maps $\mathfrak{m}_A$ into $\mathfrak{m}_B$, so it induces a map $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ as $\mathfrak{m}_A$ is killed in the composition $A \to B \to B/\mathfrak{m}_B$.

Definition 2.1. Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be noetherian local rings. A local ring homomorphism $f : A \to B$ is called unramified if $\mathfrak{m}_A B = \mathfrak{m}_B$, and $A/\mathfrak{m}_A \hookrightarrow B/\mathfrak{m}_B$ is finite and separable.

In the same situation, Let $\mathfrak{p}$ be a prime ideal in B, and let $\mathfrak{q} = f^{-1}(\mathfrak{p})$. We have the induced map $A_\mathfrak{q} \to B_\mathfrak{p}$, since if $a \notin \mathfrak{q}$, then $f(a)$ is not in $\mathfrak{p}$, so the multiplicative set $A - \mathfrak{q}$ is mapped into units in $B_\mathfrak{p}$. Is this map also unramified? No one knows when there is no finiteness assumption.

3. Etale Morphisms

Definition 3.1. Let $f : X \to Y$ be a morphism of scheme. We say it is etale if f is locally of finite presentation (pf), flat, and unramified.

For unramified, if $f(x) = y$, we have a map of local rings $f^\# : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, and we require $f^\#(\mathfrak{m}_y)\mathcal{O}_{X,x} = \mathfrak{m}_x$, and the residue field extension $k(y) \hookrightarrow k(x)$ is a finite separable extension.

Lemma 3.2. *For $f : X \to Y$ locally finite presented and flat, the following are equivalent:*

(1) $\Omega^1_{X/Y} = 0$;

(2) *f is unramified, hence etale;*

(3) *f is smooth of relative dimension 0;*

(4) *f is formally etale: given $I \subset A$ is a nilpotent ideal and a diagram*

$$\begin{tikzcd} \operatorname{Spec} A/I \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is a unique way to filling the dashed arrow. (Think of it as saying f is a local diffeomorphism.)

(5) *f is locally standard etale: for each $x \in X$ and $y = f(x)$, there exists affine neighborhoods $U = \operatorname{Spec} S \ni x$ and $V = \operatorname{Spec} R \ni y$, where S is of the form $S = R[x]_{h/g}$ where g is monic and g' is a unit in $R[x]_h$.*

Here are some examples.

(1) Let E be an abelian variety over a field k. The multiplication map $[n] : E \to E$ where $(n, \operatorname{char}(k)) = 1$ is etale.

(2) $\mathbf{G}_m \to \mathbf{G}_m$ given by $t \to t^n$ is etale if $(n, \operatorname{char}(k)) = 1$. This is because
$$\Omega^1_{X/Y} = k[t, t^{-1}]dt/(nt^{n-1}) = 0.$$

(3) Any open immersion is etale. So etale maps are not necessarily proper.

(4) $\mathbf{G}_m - \{1\} \to \mathbf{G}_m$ given by $t \to t^2$ is etale as long as $\operatorname{char}(k) \neq 2$ and surjective, but this is not proper.

(5) Let L/K be a finite separable extension. Then $\operatorname{Spec} L \to \operatorname{Spec} K$ is etale. Conversely, if L/K is not finite separable, then $\operatorname{Spec} L \to \operatorname{Spec} K$ is not etale.

(6) The Frobenius is never etale.

Lemma 3.3. *The following properties hold:*

(1) *Open immersions are etale;*

(2) *Base change of an etale maps is etale;*

(3) *A composition of etale maps is etale;*

(4) *$X \xrightarrow{f} Y \xrightarrow{g} Z$ is etale and g is etale imply f is also etale.*

These would formally imply that being etale is closed under taking products, and passing to the reduced subscheme.

4. Sites and Sheaves

Definition 4.1. Let C be a category. A Grothendieck topology on C is the following data: for each object X, it assigns a set $\operatorname{Cov}(X)$ of collections of morphisms $\{X_i \to X\}_{i \in I}$ such that

(1) it contains all isomorphisms: if $V \to X$ is an isomorphism, then $\{V \to X\} \in \operatorname{Cov}(X)$;

(2) if $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is any morphism in C, then the fiber products $X_i \times_X Y$ exists in C and the collection
$$\{X_i \times_X Y \to Y\}_{i\in I}$$
is in $\mathrm{Cov}(Y)$;

(3) if $\{X_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$, and if for every $i \in I$ we are given $\{V_{ij} \to X_i\}_{j\in J_i} \in \mathrm{Cov}(X_i)$, then the collection of composition
$$\{V_{ij} \to X_i \to X\}_{i\in I, j\in J_i}$$
is in $\mathrm{Cov}(X)$.

A category with a Grothendieck topology is called a site.

Elements in $\mathrm{Cov}(X)$ are called coverings of X.

As an example, let X be a topological space and let C be the category of open sets, where morphisms are inclusions. Then for each open subset $U \subset X$, we can let $\mathrm{Cov}(U)$ to be the set of all open covers of U, i.e. $\{V_i \to U\}_{i\in I}$ is in $\mathrm{Cov}(U)$ if and only if V_i are all open subsets of U and $U = \cup_{i\in I} V_i$. Then condition 1 is satisfied. For consition, if $W \to U$ is an inclusion of open subsets, and $\{V_i\}$ is an open cover of U, then $V_i \times_U W$, which is just $V_i \cap W$, form an open cover of W. Lastly, if $\{V_i\}_{i\in I}$ covers U and $\{W_{ij}\}_{j\in J_i}$ covers V_i, then $\{W_{ij}\}_{i\in I, j\in J_i}$ is an open cover of U. Thus this is a Grothendieck topology on the category of open sets for a topological space X.

Let S be a scheme. Let C be the category of S-schemes. For an object X in this category, define $\mathrm{Cov}(X)$ to be the set of collections of S-morphisms $\{U_i \to X\}_{i\in I}$ such that each $U_i \to X$ is an open immersion and $X = \cup_{i\in I} U_i$. Clearly condition 1 is met since isomorphisms are open immersions. For condition 2, suppose $\{U_i \to X\}_{i\in I} \in \mathrm{Cov}(X)$ and $f : Y \to X$ is any S-morphism, then $U_i \times_X Y \to Y$ are open immersions and they cover Y (these are just $f^{-1}(U_i)$). Finally the third condition is clearly true. This is the big Zariski site of a scheme.

Let X be a scheme. Let C be the category whose objects are etale morphisms $U \to X$ (namely it is a full subcategory of the category of X-schemes, where the structure map is required to be etale.) Note that a morphism in this category is a diagram

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

Here the composition is etale, and the second step is etale, so by Lemma 3.3 the map $U \to V$ is also etale. In particular, by Lemma 3.3 again fiber products exist in this category. So all morphisms in the category C are etale morphisms of schemes. We define a collection of morphisms $\{U_i \to U\}_{i\in I}$ to be in $\mathrm{Cov}(U)$ if the map
$$\coprod_{i\in I} U_i \to U$$
is surjective. Condition 1 is clearly true. Condition 2 is true because surjectivity is preserved under base change for arbitrary scheme morphisms. (Curiously, in proving this one needs to use the fact that for any scheme map $X \to S$ and $Y \to S$, the natural map on topological spaces $|X \times_S Y| \to |X| \times_{|S|} |Y|$ is always surjective.) Condition 3 is also clear. This is the small etale site.

Switching from a single scheme to the category of X-schemes we get the big etale site. Namely, for an X-scheme U, we define $\mathrm{Cov}(U)$ as the set of collections $\{U_i \to U\}_{i\in I}$ of X-morphisms where each $U_i \to U$ is etale and the map

$$\coprod_{i\in I} U_i \to U$$

is surjective.

Similarly, take the underlying category to be the category of X-schemes, we can also define $\mathrm{Cov}(U)$ to be the set of collections $\{U_i \to U\}_{i\in I}$ of X-morphisms where each $U_i \to U$ is flat and locally of finite presentation and the map

$$\coprod_{i\in I} U_i \to U$$

is surjective. This is the fppf site. Recall that a morphism of schemes is faithfully flat if it is flat and surjective. Also recall that a fppf morphism is (universally) open.

Definition 4.2. A presheaf on a category C is a contravariant functor from C to the category of sets. If C has a Grothendieck topology, a sheaf is a presheaf F such that for each object U in C and each of its coverings $\{U_i \to U\}_{i\in I}$, the equalizer sequence

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times U_j)$$

is exact (this in particular means the first map is injective).

The key result we want to get to is faithfully flat descent. Here is the ring theoretic descent lemma:

Lemma 4.3. *Let $f : A \to B$ be a faithfully flat ring homomorphism and let M be an A-module. Then the sequence*

$$\begin{aligned} 0 \to M \to M \otimes_A B &\to M \otimes_A B \otimes_A B \\ m \otimes b &\mapsto m \otimes 1 \otimes b - m \otimes b \otimes 1 \end{aligned}$$

is exact. In other words we have an equalizer sequence

$$0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B$$

Proof. Suppose the map $f : A \to B$ has an A-linear left inverse $g : B \to A$ (i.e. $g \circ f = id_A$). Then $M \to M \otimes_A B$ is injective since it also has a left inverse. In this case, g induces a map $g_M : M \otimes_A B \to M$, and $g_B : B \otimes B \to B$ which on pure tensors is given by $b \otimes b' \mapsto b \otimes f(g(b))$. If an element $\alpha \in M \otimes_A B$ is in the kernel, then

$$\alpha = (g_B \circ p_2)(\alpha) = (g_B \circ p_1)(\alpha) = (f_M \circ g_M)(\alpha)$$

(If $\alpha = m \otimes b$ is a pure tensor, this is saying $m \otimes b = m \otimes f(g(b))$.) Thus we get the desired exactness everywhere. Note that this doesn't even use faithful flatness.

Now. by faithful flatness, the exactness of such our sequence is equivalent to the exactness of the sequence after tensoring by B. But in that case thinking of B as A and $B \otimes_A B$ as B, we do have a left inverse $B \otimes_A B \to B$ given by multiplication. So we are done. ∎

Corollary 4.4. *Let X be an affine scheme. If $V \to U$ is a faithfully flat map of affine schemes, then the sequence*

$$h_X(U) \to h_X(V) \rightrightarrows h_X(V \times_U V)$$

is exact.

Proof. Let $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$, and $X = \operatorname{Spec} R$. Faithful flatness implies
$$0 \to A \to B \to B \otimes_A B$$
is exact by the previous lemma. And $\operatorname{Hom}(R, -)$ is left exact, so we are done. ∎

We develop criteria for presheaves to be fppf sheaves.

Lemma 4.5. *Let F be a presheaf on the category of schemes which is a Big Zariski sheaf. Then F is a sheaf for the fppf topology if and only if for every fppf morphism $V \to U$, the sequence*
$$F(U) \to F(V) \rightrightarrows F(V \times_U V)$$
is exact. In fact, it suffices to check only for affine schemes U and V.

Proof. First, any fppf morphism $V \to U$ is surjective, so $\{V \to U\}$ is a covering of U in the fppf topology, so if F is a fppf sheaf then this condition is trivially true.

Take a general fppf covering $\{U_i \to U\}_{i \in I}$. Let $V = \coprod_{i \in I} U_i$. Then $\{U_i\}$ is an open cover of V in the Zariski topology whose elements are disjoint. Since F is a big Zariski sheaf, the sequence
$$F(V) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j) = 0$$
is exact, so $F(V) \cong \prod_i F(U_i)$. Similarly, $\{U_i \times_U U_j\}_{i,j \in I}$ is an open cover of $V \times_U V$ in the Zariski topology. So the sequence
$$0 \to F(V \times_U V) \to \prod_{i,j} F(U_i \times_U U_j) \rightrightarrows 0$$
is exact (the third term contains elements of the form $(U_i \times_U U_j) \cap (U_k \times U_l)$ which is empty. If there is only one U_i, then the equalizer is trivial so the sequence is still exact.) Thus $F(V \times V) \cong \prod_{i,j} F(U_i \times_U U_j)$ is an isomorphism. By functoriality we have a commutative diagram

$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d] & F(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & F(V \times V) \arrow[d] \\
F(U) \arrow[r] & \prod_{i \in I} F(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} F(U_i \times_U U_j)
\end{tikzcd}$$

where the top row is exact by assumption and the vertical maps are isomorphisms, so the bottom row is also exact.

Now only assume the condition is true for affine U and V. We just need to show this implies the same condition for all schemes U and V. So take an arbitrary fppf morphism $f : V \to U$. We need to show $F(U) \to F(V)$ is injective and its image is the equalizer of $F(V) \rightrightarrows F(V \times_U V)$. Let $U = \cup_i U_i$ be an affine open cover, let $V_i = f^{-1}(U_i)$, and choose an affine open cover $V_i = \cup_j V_{ij}$ for each V_i. Each V_{ij} maps into U_i, so we have a commutative diagram

$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d] & F(V) \arrow[d] \\
\prod_{i \in I} F(U_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} F(V_{ij})
\end{tikzcd}$$

Since F is a Zariski sheaf, the vertical maps are injections. Also, the image of each V_{ij} inside U_i is open because fppf morphisms are open, so these images form an open cover of U_i. U_i is affine so it is quasi-compact, so we can choose a finite subcover $\{V_{ij_s} \to U_i\}_{s=1}^r$. This is an fppf covering of U_i where everything is affine, so by assumption

$$F(U_i) \to \prod_{s=1}^{r} F(V_{ij_s})$$

is injective (the assumption is used on the disjoint union of all the V_{ij_s}). Taking the product over all i, we see that the above diagram the bottom map is also injective, so the top arrow is also injective.

Now we need to show the exactness of

$$F(U) \to F(V) \rightrightarrows F(V \times_U V)$$

in the middle. We need to reduce to the case where U is affine and V is quasi-compact. Let $U = \cup_i U_i$ be an affine open cover, and let V_i be the preimage of U_i. If we know the statement is true for affines, then each

$$F(U_i) \to F(V_i) \rightrightarrows F(V_i \times_{U_i} V_i)$$

is exact. Then using the commutative diagram formed by various sheaf properties

$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d, "a"'] & F(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "b"'] & F(V \times_U V) \arrow[d] \\
\prod_{i\in I} F(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i F(V_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_{i,j} F(V_i \times_{U_i} V_i) \\
\prod_{i,j} F(U_i \cap U_j) \arrow[r, "c"] & \prod_{i,j} F(V_i \cap V_j) &
\end{tikzcd}$$

we will get that the top row is exact. Notice that the map c is injective since $V_i \cap V_j \to U_i \cap U_j$ is an fppf covering and we have shown injectivity.

The reduction to V quasi-compact seems a bit technical so I'd like to skip. Now we handle the case where U is affine and V is quasi-compact. Let $V = \cup_j V_j$ be a finite affine open cover, by quasi-compactness. Then $\coprod_j V_j$ is affine, and $\coprod_j V_j \to V \to U$ is an fppf covering. Now the assumption gives the exactness of the bottom row in the diagram below

$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d] & F(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, hook] & F(V \times_U V) \arrow[d, hook] \\
F(U) \arrow[r] & F(\coprod_i V_i) \arrow[r, shift left] \arrow[r, shift right] & F(\coprod_i V_i \times_U \coprod_i V_i)
\end{tikzcd}$$

and we have shown the injectivity of the vertical maps. This implies the top row is exact in the middle. ∎

We are ready to prove the main theorem.

Theorem 4.6. *Let Y be any scheme. For any Y-scheme X, the functor $h_X = \mathrm{Hom}_Y(-, X)$ is a sheaf in the fppf topology on the category of Y-schemes.*

Proof. First notice that Y doesn't play any role: if we know the result for $Y = \operatorname{Spec} \mathbf{Z}$, then we know it for any Y. Also, we have already proven the case where X is affine by using the previous criterion and the ring theoretic descent lemma.

So let X be any scheme and let $X = \cup_i X_i$ be an affine open cover. Using the previous criterion, we just need to take an fppf morphism $t : U \to V$ where U, V are affine, and check that

$$h_X(U) \to h_X(V) \rightrightarrows h_X(V \times_U V)$$

is exact.

First we prove injectivity of the first map. Let $f, g : U \to X$ be morphisms such that $t \circ f = t \circ g$. Since t is fppf it is surjective, so f, g must be set-theoretically equal. Therefore $f^{-1}(X_i) = g^{-1}(X_i)$, which we denote by U_i, and by the affine result $f|_{U_i} = g|_{U_i}$ as morphisms of schemes. Thus $f = g$ as morphisms of schemes.

Next we check exactness in the middle. Let $f : V \to X$ be a moprhism such that $f \circ p_1 = f \circ p_2$ where $p_1, p_2 : V \times_U V \rightrightarrows V$ are the two projections. Now is the moment where use the fact that on topological spaces $|V \times_U V| \to |V| \times_{|U|} |V|$ is surjective. So if we denote by $\pi_1, \pi_2 : |V| \times_{|U|} |V| \to |V|$ the two topological projections, we see that $f \circ \pi_1 = f \circ p_2$ because they are equal after composing by that surjection on the left. Now for any element $u \in U$, since the map t is surjective we take some preimage $v \in V$, and consider $f(v)$. If v' is another preimage, that means (v, v') is an element in $|V| \times_{|U|} |V|$, so then $f(v) = f(v')$. Hence this procedure is a well-define way to give a set map $h : U \to X$. Furthermore, h is continuous: if W is any open set in X, we have $h^{-1}(W) = t(f^{-1}(W))$, and t is fppf so it is open. Finally this continuous map is clearly unique with its properties. This argument can be summarized by a coequalizer diagram

$$\begin{tikzcd}
|V \times_U V| \arrow[r, two heads] & |V| \times_{|U|} |V| \arrow[r, shift left] \arrow[r, shift right] & |V| \arrow[r, "f"] \arrow[d, "t"'] & |X| \\
 & & |U| \arrow[ur, dashed, "h"'] &
\end{tikzcd}$$

realizing $|U|$ as the coequalizer of $|V \times_U V| \rightrightarrows |V|$ in the category of topological spaces.

We need to upgrade $h : U \to X$ to a map of schemes. Let $V_i = f^{-1}(X_i)$ and $U_i = h^{-1}(X_i)$. Then $t|_{V_i} : V_i \to U_i$ is fppf, and give the same after composing with the two projections. By the affine result of exactness, there exist unique morphisms of schemes $h_i : U_i \to X_i$ such that $f|_{V_i} = h_i \circ t|_{V_i}$. To see h_i and h_j agree on $X_i \cap X_j$, we can cover the intersection by affine opens, apply the same affine exactness result, and conclude by uniqueness. Hence we obtain a global map of schemes $U \to X$, and it must agree with the h we defined previously by the uniqueness of the topological map. So we are done. ■