MORE ON SHEAVES

NICOLÁS VILCHES

Abstract. We continue the discussion from last week, specializing to sheaves on the étale sites. We will mostly cover [Mil80, Chapter 3].

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1. Sheaves and presheaves

Let us recall that a *site* $\mathcal{S}$ is a category endowed with a Grothendieck topology. We also introduced the notion of a *sheaf*: a presheaf $F\colon \mathcal{S} \to \mathrm{Set}^{\mathrm{op}}$ satisfying the sheaf condition.

Definition 1. Given a site $\mathcal{S}$, we denote by $\mathrm{PSh}(\mathcal{S})$ (resp. $\mathrm{Sh}(\mathcal{S})$) the category of presheaves (resp. sheaves) of *abelian groups* $\mathcal{S}$.

We point out that our notation is slightly different from [Mil80]; our notation follows [Stacks, Tag 00UZ]. We also point out that we are specializing to (pre)sheaves of abelian groups on $\mathcal{S}$.

Lemma 2 ([Mil80, p. 48])**.** *For any site $\mathcal{S}$, the category $\mathrm{PSh}(\mathcal{S})$ is abelian.*

Proof. The same proof for sheaves on topological spaces works here. The key point is that $F \to G \to H$ is exact if and only if for any $U \in \mathcal{S}$, the sequence $F(U) \to G(U) \to H(U)$ is exact. This reduces all the constructions (kernels, cokernels and such) to the case of abelian groups. □

It is natural to ask ourselves whether the same holds for the category of sheaves on $\mathcal{S}$. If we remember our proof for topological spaces, the proof requires to construct a *sheafification* functor, that given a presheaf F constructs a "most efficient" sheaf $F^\sharp$. Let us put this into words.

Lemma 3 ([Mil80, pp. 62–3])**.** *The inclusion functor $i\colon \mathrm{Sh}(\mathcal{S}) \to \mathrm{PSh}(\mathcal{S})$ admits a left adjoint $\sharp\colon \mathrm{PSh}(\mathcal{S}) \to \mathrm{Sh}(\mathcal{S})$.*

Before we proceed with the proof, let us briefly recall the construction for sheaves on topological spaces. Given a presheaf F on a topological space X, we set

$$F^\sharp(U) = \{(s_p \in F|_p)_{p\in U} : \text{locally } s_p = t|_p, t \in F(V)\}.$$

The big issue in our case is that for a general site, we do not have a well defined notion of stalks $F|_p$. Even if we are dealing with a "geometric" site (such as the small étale site), the category of "sheaves over $\operatorname{Spec} k$" is more subtle than the usual Zariski version[1]. This way, a slightly different construction must be taken.

Proof. (Sketch) Given a presheaf F, we set F^+ to be the presheaf given by "locally defined sections":

$$F^+(U) = \lim_{\{V_i\to U\}\in \operatorname{Cov}(U)} \ker\left(\prod_i F(V_i) \rightrightarrows \prod_{i,j} F(V_i \times_U V_j)\right).$$

Note that if $s \in F^+(U)$ is a section that is zero in a cover, then $s = 0$. But we cannot glue sections in general. The problem is that this (eventually) reduces to having sections $s, s' \in F(V)$ with $s|_{V_i} = s'|_{V_i}$ for an open cover $\{V_i \to V\}$. But this does not imply that $s = s'$.

However, applying the construction *twice* fixes it: we get a well-defined functor $F \mapsto F^{++} = F^\sharp$. Checking that this is an adjoint is standard. □

Corollary 4. *The category* $\operatorname{Sh}(\mathcal{S})$ *is abelian. The functor* $i\colon \operatorname{Sh}(\mathcal{S}) \to \operatorname{PSh}(\mathcal{S})$ *is left exact, while* $\sharp\colon \operatorname{PSh}(\mathcal{S}) \to \operatorname{Sh}(\mathcal{S})$ *is exact.*

Proof. The first part follows the usual trick for sheaves of topological spaces: run everything on presheaves and take the sheafification if needed. The left exactness of i and right exactness of $\sharp$ are formal from adjunction. At last, the left exactness of $\sharp$ can be checked by noting that $F \mapsto F^+$ is left exact, which is clear from the definition. □

Example 5. Let us revisit one of our simplest examples: the constant sheaf. Given a topological space X, the constant presheaf $\mathbb{Z}$ is just given by the recipe $\mathbb{Z}(U) = \mathbb{Z}$. Its sheafification is the constant sheaf $\underline{\mathbb{Z}}$. One quickly checks that $\underline{\mathbb{Z}}(U)$ is the set of continuous functions $U \to \mathbb{Z}$, where $\mathbb{Z}$ is endowed with the discrete topology[2].

In particular, if X is a scheme, we get the constant sheaf $\underline{\mathbb{Z}}$ for the Zariski topology. Now, if X is irreducible, then $\mathbb{Z}$ is already a sheaf in the Zariski topology, so this seems like an uninteresting sheaf. Moreover, we have that $\mathbb{Z}$ is flasque, hence all $H^i(X, \mathbb{Z})$ vanish.

On the other hand, we can consider the constant presheaf $U \mapsto \mathbb{Z}$ as a presheaf on the small étale site, to get an étale sheaf $\underline{\mathbb{Z}}_{\text{ét}} \in \operatorname{Sh}(X_{\text{ét}})$. This is no longer flasque! In fact, we will see later that its cohomology is extremely interesting.

2. Pullbacks and pushforwards

Our next goal is to make sense of pullbacks and pushforwards in various contexts. Let us start with the following definition.

Definition 6. Let $\mathcal{S}$ and $\mathcal{T}$ be sites. A functor $u\colon \mathcal{S} \to \mathcal{T}$ is *continuous* if it preserves coverings and fibered products.

[1]If k is not separably closed, then $\operatorname{Spec} k$ admits interesting étale covers!

[2]I informally call this the "locally constant sheaf", but this is non-standard.

Example 7. There are three key examples to have in mind.

(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. The preimage map $V \subset Y \mapsto f^{-1}(V) \subset X$ defines a functor $u\colon \mathrm{Op}(Y) \to \mathrm{Op}(X)$. It is easy to see that this functor is continuous: if $\{V_i \subset V\}$ is an open cover of some $V \subset X$, then $\{f^{-1}(V_i) \subset f^{-1}(V)\}$ is an open cover.
(2) Similarly, say that $f\colon X \to Y$ is a morphism of schemes. The previous construction gives us a functor $Y_{\mathrm{Zar}} \to X_{\mathrm{Zar}}$ between the small Zariski sites. A similar construction (replacing the preimage with the fibered product) gives us a functor $Y_{\mathrm{Zar}} \to X_{\mathrm{Zar}}$ between the small étale sites, and something similar for the fppf/fpqc sites.
(3) Let X be a scheme. Note that every open immersion map $U \to X$ is étale. This way, we have a well-defined functor $X_{\mathrm{Zar}} \to X_{\text{ét}}$, which is clearly continuous. There is a similar construction for the fppf/fpqc topologies.

To motivate the constructions, it is a good idea to keep the first example in mind; my impression is that it is the most concrete one.

Given a continuous functor $u\colon \mathcal{S} \to \mathcal{T}$, there are a couple of constructions that we can perform. First, we can define a *pushforward* functor $u_p\colon \mathrm{PSh}(\mathcal{T}) \to \mathrm{PSh}(\mathcal{S})$ (note the change on the order!), via the formula

$$u_p(G)(U) = G(u(U)).$$

It is immediate that this functor is exact. The *pullback* functor $u^p\colon \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}(\mathcal{T})$ is a bit trickier to define.

Lemma 8 ([Mil80, p. 59])**.** *The functor $u_p\colon \mathrm{PSh}(\mathcal{T}) \to \mathrm{PSh}(\mathcal{S})$ admits a left adjoint $u^p\colon \mathrm{PSh}(\mathcal{S}) \to \mathrm{PSh}(\mathcal{T})$. It is exact if finite inverse limits exists in $\mathcal{S}$.*

Proof. (Idea) Fix $F \in \mathrm{PSh}(\mathcal{S})$. For $V \in \mathcal{T}$, we define $(u^pG)(V) = \mathrm{colim}_{(U,\phi)} G(U)$. The colimit is taken over "covers of V": pairs $\phi\colon V \to \phi(U)$. □

Example 9. Let $f\colon X \to Y$ be a continuous map between topological spaces. If $u\colon \mathrm{Op}(Y) \to \mathrm{Op}(X)$ is the inverse image map, we get that $u_p = f_*\colon \mathrm{PSh}(X) \to \mathrm{PSh}(Y)$, and $u^p = f^{-1}\colon \mathrm{PSh}(Y) \to \mathrm{PSh}(X)$.

Definition 10 ([Mil80, p. 68])**.** Let $u\colon \mathcal{S} \to \mathcal{T}$ be a continuous functor. Define $u_*\colon \mathrm{Sh}(\mathcal{T}) \to \mathrm{Sh}(\mathcal{S})$ as the restriction of u_p, and $u^*\colon \mathrm{Sh}(\mathcal{S}) \to \mathrm{Sh}(\mathcal{T})$ as the sheafification of u^p.

3. Sheaves on the small étale site

Given a scheme X, we let $X_{\text{ét}}$ be the small étale site on X, and $\mathrm{Sh}(X_{\text{ét}})$ the category of sheaves on this site. Our goal for this section is to describe basic properties of this category.

First of all, let us specialize the discussion from the previous section to this setup. If $f\colon X \to Y$ is a morphism of schemes, we get an induced morphism $u\colon Y_{\text{ét}} \to X_{\text{ét}}$. This way, we get functors $f_*\colon \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}})$ and $f^*\colon \mathrm{Sh}(Y_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$.

In particular, if $i\colon \overline{x} \to X$ is a geometric point, we get a restriction functor $i^*\colon \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(\overline{x}_{\text{ét}})$. Note that the right hand side is equivalent to the category of abelian groups, as $\kappa(\overline{x})$ is separably closed.

Lemma 11 ([Mil80, 2.10])**.** *Let $F \in \mathrm{Sh}(X_{\text{ét}})$, and let $s \in F(U)$ be non-zero. Then there exists a geometric point $\overline{x} \to U$ such that $s|_{\overline{x}} \neq 0$.*

Proof. We argue by contradiction: assume that for each geometric point $\overline{x} \to U$, $s|_{\overline{x}} = 0$. This way, for each point $u \in U$, pick a geometric point $\overline{u} \to U$. By definition, the condition $s|_{\overline{u}} = 0$ implies that there exists an étale map $V_u \to U$ containing u on its image, such that $s|_{V_u} = 0$. Put the V_u together to get an étale cover. □

Proposition 12 ([Mil80, 2.15])**.** *Let X be a scheme.*

1. *A sequence $0 \to F \to F' \to F''$ on $\mathrm{Sh}(X_{\text{ét}})$ is left exact if and only if for every étale map $U \to X$, $0 \to F(U) \to F'(U) \to F''(U)$ is left exact, if and only if for every geometric point $\overline{x} \to X$, the sequence $0 \to F|_{\overline{x}} \to F'|_{\overline{x}} \to F''|_{\overline{x}}$ is left exact.*
2. *A sequence $F \to F' \to F'' \to 0$ on $\mathrm{Sh}(X_{\text{ét}})$ is right exact if and only if for every geometric point, the sequence $F|_{\overline{x}} \to F'|_{\overline{x}} \to F''|_{\overline{x}} \to 0$ is right exact.*

Proof. (1) The first equivalence follows directly by adjunction. The fact that this implies exactness on $\overline{x}$ follows from the exactness of i^*.

Now, assume that for every geometric point $\overline{x} \to X$, the sequence on stalks is left exact. Let us first show that $F(U) \to F'(U)$ is injective for any étale map $U \to X$. Fix $s \in \ker(F(U) \to F'(U))$. If $\overline{x} \to U$ is a geometric point, we have that $s|_{\overline{x}} \in \ker(F|_{\overline{x}} \to F'|_{\overline{x}})$, and so $s|_{\overline{x}} = 0$. It follows that $s = 0$ by the previous lemma.

Similarly, let $s \in \ker(F'(U) \to F''(U))$. For each $u \in U$, we get that $s|_{\overline{u}} \in F|_{\overline{u}}$. This way, there exists an étale $V_u \to U$ such that $s|_{V_u}$ lies in the image of $F(V_u)$. The sheaf property allows us to conclude.

(2) Omitted. □

Let us compare this to the usual statement for sheaves on the small Zariski site; instead of checking exactness on *points*, we check it on *geometric points*. This is extremely useful, as geometric points are separably closed. This will help us constructing unexpected short exact sequences.

Example 13 ([Mil80, 2.18(b)])**.** Fix a scheme X over a field of characteristic zero. Consider the sheaf $\mathbb{G}_m \in \mathrm{Sh}(X_{\text{ét}})$ given by $\mathbb{G}_m(U) = \Gamma(U, \mathscr{O}_U)^\times$. Note that this agrees with $\mathrm{Hom}(U, \operatorname{Spec} \mathbb{Z}[T, T^{-1}])$, and so it is a sheaf for the étale topology.

Now, consider the "multiplication by n" morphism $\times n \colon \mathbb{G}_m \to \mathbb{G}_m$. It is clear that its kernel corresponds to μ_n, the sheaf whose sections over U are nth roots of unity. This way, we have a sequence $0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m$ on $\mathrm{Sh}(X_{\text{ét}})$. So far, everything we have said could've been done in the Zariski site.

We claim that $\mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m$ is surjective. (Note that the analogous statement is false in the Zariski topology! Not every function is Zariski-locally an nth root.) To prove this, it suffices to prove it for a separably closed field k, thanks to the previous proposition. But in this case the result is clear! This way, we get the short exact sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \to 0$$

on the étale site (even though the analogous sequence is not exact on the Zariski site!)

Example 14 ([Mil80, 2.18(c)])**.** If X is a scheme over a field of characteristic p, we can construct a similar story with $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \to \mathbb{G}_a \to 0$, where the right map is $F - 1$. This is known as the *Artin–Schreier sequence*.

Example 15 ([Mil80, pp. 52–3])**.** Let us briefly discuss the situation over fields carefully. Given a field k, we let $X = \operatorname{Spec} k$. As we have mentioned before, if k is separably closed, its étale site is non-interesting, and $\mathrm{Sh}(X) \cong \mathrm{Ab}$.

However, when k is not separably closed things become much more interesting. For example, say that we want to study $\operatorname{Spec}\mathbb{R}$. There is a single interesting covering: $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$. This way, the data of a sheaf over $\operatorname{Spec}\mathbb{R}$ consists on two abelian groups, $M_{\mathbb{R}}$ and $M_{\mathbb{C}}$, satisfying some compatibility relation. This relation can be obtained by spelling out the sheaf condition over $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$: we need

$$0 \to M_{\mathbb{R}} \to M_{\mathbb{C}} \to M(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$$

to be exact. Now, the last space can be identified with $M(\mathbb{C}) \oplus M(\mathbb{C})$, so that the two maps from the equalizer are $s \mapsto (s, s)$ and $s \mapsto (s, \sigma(s))$, where $\sigma \colon M_{\mathbb{C}} \to M_{\mathbb{C}}$ is induced by the involution on $\mathbb{C}$. In other words, the data of a $\operatorname{Spec}\mathbb{R}$-sheaf corresponds to an abelian group $M = M_{\mathbb{C}}$, plus a (left) $\mathbb{Z}/2\mathbb{Z}$ action; in this case, $M_{\mathbb{R}}$ is the fixed part of σ.

Proposition 16 ([Mil80, 1.9])**.** *Let k be a field. Denote by $G = \mathrm{Gal}(k_{sep}/k)$ its Galois group, endowed with the pro-finite topology. The category $\mathrm{Sh}(X_{\text{ét}})$ is equivalent to the category of discrete G-modules.*

We omit the proof. In any case, the idea is basically what we did before. The only issue is that in general k_{sep}/k is not finitely generated; thus, we cannot evaluate $M_{k_{sep}}$. Instead, we can take a limit of the $M_{k'}$ for finite separable extensions k'/k.

References

[Mil80] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. No. 33, Princeton University Press, Princeton, NJ, 1980. MR0559531

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Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027, USA

Email address: `nivilches@math.columbia.edu`